REPRESENTATION HOMOLOGY
AND
KNOT CONTACT HOMOLOGY

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This thesis has four parts. In the first part, we introduce and study representation homology of topological spaces, which is a higher homological extension of representation varieties of fundamental groups. We give an elementary construction of representation homology in terms of classical (abelian) homological algebra. Our construction is parallel to Pirashvili's construction of higher Hochschild homology; in fact, we establish a direct relation between the two theories by proving that the representation homology of the (reduced) suspension of a (pointed connected) space is isomorphic to its higher Hochschild homology. We also construct some natural maps and spectral sequences relating representation homology to other known homology theories associated with spaces (such as the Pontryagin algebra $\mathrm{H}_*(\Omega X)$, the $\mathbb{S}^1$-equivariant homology of the free loop space and the stable homology of automorphism groups of f.g. free groups). We compute representation homology explicitly in a number of interesting cases, including the spheres $\mathbb{S}^n$, the complex projective spaces $\mathbb{CP}^r$, closed surfaces of arbitrary genus and some 3-dimensional manifolds, such as link complements in $\mathbb{R}^3$ and lens space $L(p, q)$. One of our main results, which we call Comparison Theorem, expresses the representation homology of a simply-connected topological space of finite rational type in terms of its Quillen and Sullivan models.

The second part is a compendium of the first part. We prove some technical results required to establish some basic properties of representation homology. A result that might be of independent interest is Theorem 10.3.2, which says that if k is a field of characteristic zero, then for any k-linear operad $\mathcal{P}$, the model category of simplicial $\mathcal{P}$ algebras is Quillen equivalent to the model category of non-negatively graded DG $\mathcal{P}$ algebras. This is the result that allows us to transition between simplicial commutative algebras and commuatative DG algebras, thereby proving some results (see, e.g., Proposition 11.3.13 and 11.3.14) about smooth extensions of simplicial commutative algebras. These results would find applications in derived representations schemes, which are simplicial commutative algebras that give rise to representation homology.

In the third part, we give a new algebraic construction of knot contact homology in the sense of Ng [120]. For a link L in $\mathbb{R}^3$, we define a differential graded (DG) k-category $\tilde{\mathscr{A}}_L$ with finitely many objects, whose quasi-equivalence class is a topological invariant of L. In the case when L is a knot, the endomorphism algebra of a distinguished object of $\tilde{\mathscr{A}}_L$ coincides with the fully noncommutative knot DGA as defined by Ekholm, Etnyre, Ng and Sullivan in [54]. The input of our construction is a natural action of the braid group B_n on the category of perverse sheaves on a two-dimensional disk with singularities at n marked points, studied by Gelfand, MacPherson and Vilonen in [66]. As an application, we show that the category of finite-dimensional representations of the link k-category $\tilde{A}_L = H_0(\tilde{\mathscr{A}}_L)$ defined as the 0-th homology of $\tilde{\mathscr{A}}_L$ is equivalent to the category of perverse sheaves on $\mathbb{R}^3$ that are singular along the link L. We also obtain several generalizations of the category $\tilde{\mathscr{A}}_L$ by extending the Gelfand-MacPherson-Vilonen braid group action.

In the forth part, we generalize Keller's construction [93] of deformed n-

Calabi-Yau completions to the relative contexts. This gives a universal construction that extends any given DG functor $F : \mathcal{A} \to \mathcal{B}$ to a DG functor $\tilde{F} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$, together with a family of deformations of $\tilde{F}$ parametrized by relative negative cyclic homology classes $[\eta] \in \mathrm{HC}^{-}_{n-2}(\mathcal{B}, \mathcal{A})$. We show that, under a finiteness condition, these extensions have canonical relative n-Calabi-Yau structures in the sense of [15]. This is applied to give a construction that associates a DG category $\mathscr{A}(N, M; \Phi)$ to a pair (N, M) consisting of a manifold N and an embedded submanifold M of codimension ≥ 2, together with a trivialization Φ of the unit normal bundle of M in N. In the case when (N, M) is the pair consisting of a set of n points in the interior of the 2-dimensional disk, $\mathscr{A}(N, M; \Phi)$ is the multiplicative preprojective algebra [35] with non-central parameters. In the case when (N, M) is the pair consisting of a link L in $\mathbb{R}^3$, $\mathscr{A}(N, M; \Phi)$ is the link DG category [11] that extends the Lengendrian DG algebra [120, 121, 122, 54] of the unit conormal bundle $ST^*_L(\mathbb{R}^3) \subset ST^*(\mathbb{R}^3)$. We show that, when $M \subset N$ has codimension 2, then the category of finite dimensional modules over the 0-th homology $\mathrm{H}_0(\,\mathscr{A}(N, M; \Phi)\,)$ of this DG category is equivalent to the category of perverse sheaves on N with singularities at most along M.

Our main references are the papers [9], [10] and [167], which form Part I, III and IV of this thesis respectively.

BIOGRAPHICAL SKETCH

Wai-kit Yeung received his B.Sc. in Mathematics from The Chinese University of Hong Kong in 2011. From 2011 to 2017, he has been a doctoral candidate at the Ph.D. program of Mathematics at Cornell University.

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Representation homology of spaces and higher Hochschild homology

CHAPTER 1

INTRODUCTION

1.1 Representation varieties and representation homology

Let G be a finite-dimensional affine algebraic group defined over a field k of characteristic zero. For any (discrete) group Γ, the set of all representations of Γ in G has a natural structure of an affine k-scheme called the *representation scheme* $\mathrm{Rep}_G(\Gamma)$. The representation schemes and associated representation varieties play an important role in many areas of mathematics, most notably in representation theory and low-dimensional topology. In representation theory, the fundamental problem is to understand the structure of representations of Γ in G. One can approach this problem geometrically by studying the natural (adjoint) action of the group G on the variety $\mathrm{Rep}_G(\Gamma)$. When k is algebraically closed and Γ is finitely generated, the equivariant geometry of $\mathrm{Rep}_G(\Gamma)$ is closely related to the representation theory of Γ: the equivalence classes of representations of Γ in G are in bijection with the G-orbits in $\mathrm{Rep}_G(\Gamma)$, and the geometry of G-orbits determines the algebraic structure of representations. This relation has been extensively studied since the late 70's, and the representation varieties have become a standard tool in representation theory of groups (see, for example, [109]).

In topology, one is usually interested in global algebro-geometric invariants of spaces defined in terms of representation varieties of fundamental groups. For example, if K is a knot in $\mathbb{S}^3$, many classical invariants of K arise from its character variety $\chi_G(K) := \mathrm{Rep}_G[\pi_1(X_K)]//G$, which is the (categorical) quotient of the representation variety of the fundamental group of the knot complement $X_K := \mathbb{S}^3 \setminus K$. These invariants include, in particular, the Alexander polynomial $\Delta_K(t)$ (in the simplest case when $G = \mathbb{C}^*$, see, e.g., [111]), the so-called A-polynomial $A_K(m, l)$ (see [39]), the Casson invariant [110] and the famous Chern-Simons invariant [97, 98], all of which are defined for $G = \mathrm{SL}_2(\mathbb{C})$. In fact, for $G = \mathrm{SL}_2(\mathbb{C})$, the entire character variety, or rather its coordinate ring $\mathcal{O}[\chi_G(K)]$, has a purely topological interpretation as a Kauffman bracket skein module of X_K (see [131]).

Despite being useful tools, the representation varieties exhibit some intrinsic deficiencies. First of all, these varieties are usually very singular, which makes it hard to understand their geometry. Thus, in representaion theory, one faces the problem of resolving singularities of $\mathrm{Rep}_G(\Gamma)$. In topology, the use of representation varieties is mostly limited to (compact orientable) surfaces, hyperbolic 3-manifolds and knot complements in $\mathbb{S}^3$, all of which are known to be aspherical spaces. The homotopy type of such a space is completely determined by the isomorphism type of its fundamental group, which makes representation varieties of these groups very strong and efficient invariants. For more general spaces, however, one needs to take into account a higher homotopy information, and looking at representation varieties of fundamental groups (or even, higher homotopy groups) is not enough.

A natural way to remedy these problems is to replace the representation functor Rep_G with its (non-abelian) derived functor DRep_G much in the same way as one replaces non-exact (additive) functors in classical homological algebra (such as '$\otimes$' and 'Hom') with corresponding derived functors ('$\otimes^{\boldsymbol{L}}$' and '$\mathbf{R}\mathrm{Hom}$'). Geometrically, passing from the representation scheme $\mathrm{Rep}_G(\Gamma)$ to the derived representation scheme $\mathrm{DRep}_G(\Gamma)$ amounts to desingularizing[1] $\mathrm{Rep}_G(\Gamma)$,

[1]This approach to resolution of singularities is discussed in [40, 41], where it is applied to a

while topologically, this brings in a new homology theory of spaces that captures a good deal of homotopy information and refines the classical representation varieties of fundamental groups in an interesting and non-trivial way.

To explain this idea in more precise terms, we recall that the representation scheme $\mathrm{Rep}_G(\Gamma)$ is defined as the functor on the category of commutative k-algebras:

$$\mathrm{Rep}_G(\Gamma) : \mathtt{CommAlg}_k \to \mathtt{Set}\,, \quad A \mapsto \mathrm{Hom}_{\mathtt{Gr}}(\Gamma,\, G(A))\,, \tag{1.1.1}$$

assigning to an algebra A the set of families of representations of Γ in G parametrized by the k-scheme $\mathrm{Spec}(A)$. It is well known (and easy to prove) that the functor (1.1.1) is representable, and we denote the corresponding commutative algebra by Γ_G[2]. Varying Γ (while keeping G fixed), we can now regard Γ_G as a functor on the category of groups:

$$(\,\text{–}\,)_G : \mathtt{Gr} \to \mathtt{CommAlg}_k\,, \quad \Gamma \mapsto \Gamma_G\,, \tag{1.1.2}$$

which we call the *representation functor* in G. The functor (1.1.2) extends naturally to the category $\mathtt{sGr}$ of simplicial groups, taking values in the category $\mathtt{sCommAlg}_k$ of simplicial commutative algebras. Both categories $\mathtt{sGr}$ and $\mathtt{sCommAlg}_k$ carry standard (simplicial) model structures, with weak equivalences being the weak homotopy equivalences of the underlying simplicial sets. The functor $(\,\text{–}\,)_G : \mathtt{sGr} \to \mathtt{sCommAlg}_k$ is not homotopy invariant: it does not preserve weak equivalences in general and hence does not descend to a functor between the homotopy categories $\mathrm{Ho}(\mathtt{sGr})$ and $\mathrm{Ho}(\mathtt{sCommAlg}_k)$. However, it is easy to check that $(\,\text{–}\,)_G$ does map weak equivalences between cofibrant objects

number of classical moduli problems in algebraic geometry.

[2]The algebra Γ_G may be thought of as the coordinate ring of the affine k-scheme $\mathrm{Rep}_G(\Gamma)$, and we sometimes write $\Gamma_G = \mathcal{O}[\mathrm{Rep}_G(\Gamma)]$ to emphasize this interpretation.

in $\mathtt{sGr}$ to weak equivalences in $\mathtt{sCommAlg}_k$ (see Lemma 4.2.4). Hence, by standard homotopical algebra, it has a (total) left derived functor

$$\boldsymbol{L}(-)_G : \mathrm{Ho}(\mathtt{sGr}) \to \mathrm{Ho}(\mathtt{sCommAlg}_k)\,. \tag{1.1.3}$$

We call (1.1.3) the *derived representation functor* in G. Heuristically, $\boldsymbol{L}(-)_G$ may be thought of as a "best possible" universal approximation to the representation functor (1.1.2) at the level of homotopy categories. When applied to a simplicial group Γ, the functor (1.1.3) is represented by a simplicial commutative algebra which (abusing notation) we denote by $\mathrm{DRep}_G(\Gamma)$. The homotopy groups of $\mathrm{DRep}_G(\Gamma)$ depend only on Γ and G, with $\pi_0[\mathrm{DRep}_G(\Gamma)]$ being canonically isomorphic to $\pi_0(\Gamma)_G$. In particular, if Γ is a discrete simplicial group, then $\pi_0[\mathrm{DRep}_G(\Gamma)] \cong \Gamma_G$. Extending our terminology from [23, 24], we will refer to $\pi_*[\mathrm{DRep}_G(\Gamma)]$ as the *representation homology* of Γ in G and denote it $\mathrm{HR}_*(\Gamma, G)$. We should mention that representation homology of associative and Lie algebras was introduced and studied in [23, 21, 22, 24]. The idea of deriving the representation functor was motivated by noncommutative geometry, where the representation functor plays an important role (see [69, 101] and also [22]).

Next, we recall that the model category of simplicial groups is Quillen equivalent to the category of reduced simplicial sets, $\mathtt{sSet}_0$, which is, in turn, Quillen equivalent to the category $\mathtt{Top}_{0,*}$ of pointed connected topological spaces. These classical equivalences are given by the pairs of adjoint functors:

$$\mathbb{G} : \mathtt{sSet}_0 \rightleftarrows \mathtt{sGr} : \overline{W}\,, \qquad |-| : \mathtt{sSet}_0 \rightleftarrows \mathtt{Top}_{0,*} : \mathrm{ES}\,,$$

which we briefly review in Section 4.1. Here, we only recall that $\mathbb{G} : \mathtt{sSet}_0 \to \mathtt{sGr}$ is defined by the Kan loop group construction [89] that assigns to a reduced simplicial set $X \in \mathtt{sSet}_0$ a semi-free simplicial group $\mathbb{G}X$, which is a simplicial model of the based loop space $\Omega|X|$. Now, applying to $\mathbb{G}X$ the derived representation functor (1.1.3), we define the *representation homology* of a reduced simplicial set $X \in \mathtt{sSet}_0$ by

$$\mathrm{HR}_*(X, G) := \boldsymbol{L}_*(\mathbb{G}X)_G = \pi_*[\mathrm{DRep}_G(\mathbb{G}X)] \ . \tag{1.1.4}$$

By definition, $\mathrm{HR}_*(X, G)$ is a (non-negatively) graded commutative k-algebra that depends only on the homotopy type of X and hence is a homotopy invariant of the corresponding space $|X|$. In degree zero, we have $\mathrm{HR}_0(X, G) \cong (\pi_1(X))_G = \mathcal{O}[\mathrm{Rep}_G(\pi_1(X))]$, where $\pi_1(X)$ is the fundamental group of X. To avoid confusion, we emphasize that $\mathrm{HR}_*(X, G) \ncong \mathrm{HR}_*(\pi_1(X), G)$ in general; however, if X is a $K(\Gamma, 1)$-space (*e.g.*, $X = B\Gamma$ for a discrete group Γ), then indeed we have $\mathrm{HR}_*(X, G) \cong \mathrm{HR}_*(\Gamma, G)$, so there is no ambiguity in our notation.

The goal of Part I is threefold. First, we give an elementary construction of representation homology in terms of classical (abelian) homological algebra. Our construction is analogous to Pirashvili's construction of higher order Hochschild homology, and it provides a natural interpretation of representation homology as functor homology. This opens the way to efficient computations and places representation homology in one row with other classical invariants such as Hochschild and cyclic homology. Second, we construct spectral sequences and natural maps relating representation homology to other homology theories associated with spaces (including $H_*(\Omega X)$, the higher Hochschild homology, the $\mathbb{S}^1$-equivariant homology of the free loop space $\mathscr{L}X$ and the stable homology of the automorphism groups of f.g. free groups $\mathbb{F}_n$). Third, we compute representation homology explicitly in a number of interesting cases, including the spheres $\mathbb{S}^n$, suspensions ΣX, closed surfaces of arbitrary genus and some classical 3-dimensional spaces, such as link complements in $\mathbb{R}^3$ and lens spaces $L(p, q)$. One of our main results is the Comparison Theorem that expresses the representation homology of a simply-connected topological space of finite rational type in terms of its Quillen and Sullivan models.

1.2 Main results

We now proceed with a summary of the main results of Part I. We begin by recalling Pirashvili's definition of higher Hochschild homology [133]. Let $\mathfrak{F} \subset$ $\mathtt{Set}$ denote the full subcategory of finite sets with objects $\underline{n} := \{1, 2, \ldots, n\}$ (for $n \geq 1$) and $\underline{0} := \varnothing$. If $\mathtt{Vect}_k$ is the category of k-vector spaces, any functor F : $\mathfrak{F} \to \mathtt{Vect}_k$ with values in $\mathtt{Vect}_k$ extends naturally to a functor $\tilde{F} : \mathtt{Set} \to \mathtt{Vect}_k$ on the category of all sets (by taking the left Kan extension along the inclusion $\mathfrak{F} \hookrightarrow \mathtt{Set}$). Now, given $F : \mathfrak{F} \to \mathtt{Vect}_k$ and any simplicial set $X \in \mathtt{sSet}$, we can define a simplicial k-vector space FX by composition

$$\Delta^{\mathrm{op}} \xrightarrow{X} \mathtt{Set} \xrightarrow{\tilde{F}} \mathtt{Vect}_k \,.$$

Following [133], we denote the homotopy groups of FX by $\mathrm{HH}_*(X, F) := \pi_*(FX) = \mathrm{H}_*[N(FX)]$, where N stands for the Dold-Kan normalization functor. Any commutative k-algebra A defines a functor $\underline{A} : \mathfrak{F} \to \mathtt{Vect}_k$ by assigning to an object $\underline{n} \in \mathfrak{F}$ the n-th tensor power $A^{\otimes n}$ and to a morphism $f : \underline{n} \to \underline{m}$ the linear map $f_* : A^{\otimes n} \to A^{\otimes m},\ f_*(a_1 \otimes \ldots \otimes a_n) = b_1 \otimes \ldots \otimes b_m$, with $b_j := \prod_{i \in f^{-1}(j)} a_i$ for $j = 1, 2, \ldots, m$. The *higher Hochschild homology* of X with coefficients in A is then defined by the formula $\mathrm{HH}_*(X, A) := \mathrm{HH}_*(X, \underline{A})$. As observed in [133], if $X = \mathbb{S}^1$ is the (simplicial) circle, then $\mathrm{HH}_*(\mathbb{S}^1, A)$ coincides with the classical Hochschild homology $\mathrm{HH}_*(A)$ of the commutative algebra A. The key fact behind this construction is that the category $\mathfrak{F}$ is a PROP (i.e., a permutative category) with monoidal structure $\underline{n} \boxtimes \underline{m} = \underline{n+m}$, and the category of k-algebras over $\mathfrak{F}$ (i.e., the category of strong monoidal functors $\mathfrak{F} \to \mathtt{Vect}_k$) is equivalent (via $A \leftrightarrow \underline{A}$) to the category of commutative algebras, $\mathtt{CommAlg}_k$.

Now, to define representation homology, we replace $\mathfrak{F}$ by the category $\mathfrak{G}$ of finitely generated free groups with objects $\langle n \rangle := \mathbb{F}\langle \underline{n} \rangle$ (for $n \geq 0$) and morphisms being the arbitrary group homomorphisms. It is known that $\mathfrak{G}$ is also a PROP with monoidal structure $\langle n \rangle \boxtimes \langle m \rangle = \langle n+m \rangle$, and the category of k-algebras over $\mathfrak{G}$ is equivalent to the category of commutative Hopf k-algebras. Under this equivalence, a commutative Hopf algebra $\mathcal{H}$ corresponds to the (strong monoidal) functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathtt{Vect}_k$, $\underline{n} \mapsto \mathcal{H}^{\otimes n}$, which actually takes its values in the category of commutative algebras. As in the case of sets, the functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathtt{CommAlg}_k$ extends naturally to the category of all (based) free groups $\mathtt{FGr}$, whose objects are the free groups $\mathbb{F}\langle S \rangle$ given with a prescribed generating set S and morphisms are the arbitrary group homomorphisms.

Now, mimicking the Pirashvili construction, for a reduced simplicial set $X \in \mathtt{sSet}_0$ and a commutative Hopf algebra $\mathcal{H}$, we can consider the composition of functors

$$\Delta^{\mathrm{op}} \xrightarrow{\mathbb{G}X} \mathtt{FGr} \xrightarrow{\tilde{\underline{\mathcal{H}}}} \mathtt{CommAlg}_k \, ,$$

where $\mathbb{G}X$ is the Kan loop group construction of X and $\tilde{\underline{\mathcal{H}}}$ is the (left) Kan extension of $\underline{\mathcal{H}}$ along the inclusion $\mathfrak{G} \hookrightarrow \mathtt{FGr}$. This defines a simplicial commutative algebra $\underline{\mathcal{H}}(\mathbb{G}X)$, whose homotopy groups we denote by

$$\mathrm{HR}_*(X, \mathcal{H}) := \pi_* \underline{\mathcal{H}}(\mathbb{G}X) = \mathrm{H}_*[N(\underline{\mathcal{H}}(\mathbb{G}X))] \, . \tag{1.2.1}$$

Our main observation is that this definition is *equivalent* to our original definition of representation homology given in terms of the derived representation functor $\boldsymbol{L}(-)_G$, see (1.1.4). Precisely, (*cf.* Proposition 4.2.11)

Proposition 1.2.2. *Let G be an affine group scheme over k with coordinate ring $\mathcal{H} = \mathcal{O}(G)$. Then, for any $X \in \mathtt{sSet}_0$, there is a natural isomorphism of graded commutative* *algebras*

$$\mathrm{HR}_*(X, G) \cong \mathrm{HR}_*(X, \mathcal{O}(G)) \ .$$

Thanks to Proposition 1.2.2, we may (and will) use the notation $\mathrm{HR}_*(X, G)$ and $\mathrm{HR}_*(X, \mathcal{H})$ interchangeably, without causing confusion. Although its proof is almost immediate, Proposition 1.2.2 has a number of important implications. First, we state the following theorem, which is the main result of Section 4 (see Theorem 4.2.17).

Theorem 1.2.3. *For any $X \in \mathtt{sSet}_0$, there is a natural first quadrant spectral sequence*

$$E^2_{pq} \; = \; \mathrm{Tor}^{\mathfrak{G}}_p(\underline{\mathrm{H}}_q(\Omega X; k), \underline{\mathcal{H}}) \underset{p}{\Longrightarrow} \mathrm{HR}_n(X, \mathcal{H}) \tag{1.2.4}$$

converging to the representation homology of X.

The spectral sequence (1.2.4) relates the representation homology $\mathrm{HR}_*(X, \mathcal{H})$ of a space X to its Pontryagin algebra $\mathrm{H}_*(\Omega X; k)$. To describe the E_2-term of (1.2.4) we recall that $\mathrm{H}_*(\Omega X; k)$ has a natural structure of a graded cocommutative Hopf algebra with coproduct induced by the Alexander-Whitney diagonal and the product by the Eilenberg-Zilber map. For each $q \in \mathbb{Z}$, the assignment $\langle n \rangle \mapsto [\mathrm{H}^{\otimes n}]_q$, where $[\mathrm{H}^{\otimes n}]_q$ is the q-th graded component of the n-th tensor power of $\mathrm{H} = \mathrm{H}_*(\Omega X; k)$, defines a functor $\underline{\mathrm{H}}_q : \mathfrak{G}^{\mathrm{op}} \to \mathtt{Vect}_k$, which is the first argument of the "Tor" in (1.2.4). The "Tor" itself is the (abelian) derived functor of the tensor product $\otimes_{\mathfrak{G}}$ between covariant and contravariant $\mathtt{Vect}_k$-valued functors over the (small) category $\mathfrak{G}$. The spectral sequence (1.2.4) is a counterpart of Pirashvili's fundamental spectral sequence for higher Hochschild homology (*cf.* [133, Theorem 2.4]); however, in the case of representation homology it takes a more geometric form.

Theorem 1.2.3 has several interesting consequences. For example, if X is a $K(\Gamma, 1)$-space, the spectral sequence (1.2.4) degenerates giving an isomorphism (*cf.* Corollary 4.2.19)

$$\mathrm{HR}_*(\Gamma, G) \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \mathcal{O}(G)) \,, \tag{1.2.5}$$

where $k[\Gamma]$ is the group algebra of Γ equipped with the natural (cocommutative) Hopf algebra structure. The isomorphism (1.2.5) shows that the representation homology of (discrete) groups has a natural 'Tor' interpretation, similar to the classical (Connes) interpretation of the Hochschild and cyclic homology (see [106, Chap. 6]). Furthermore, combined with a standard comparison theorem for spectral sequences, Theorem 1.2.3 implies that the representation homology $\mathrm{HR}_*(X, \mathcal{H})$ is stable under Pontryagin equivalences[3], and hence, if X is simply connected, then $\mathrm{HR}_*(X, \mathcal{H})$ is actually a *rational* homotopy invariant of X (*cf.* Proposition 4.2.23).

Next, in Section 5, we show that representation homology is not only analogous to higher Hochschild homology but actually related to it in a remarkably simple, geometric way. The main result of this section reads (*cf.* Theorem 5.1.1 and Theorem 5.1.2):

Theorem 1.2.6. *Let $\mathcal{H}$ be a commutative Hopf algebra.*

(a) *For any simplicial set $X \in$ `sSet`, there is a natural isomorphism*

$$\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \;\cong\; \mathrm{HH}_*(X, \mathcal{H}) \,, \tag{1.2.7}$$

where $X_+ = X \sqcup \{*\}$ is a pointed simplicial set obtained from X by adjoining functorially a basepoint $*$, and Σ is the (reduced) suspension functor on the category of pointed simplicial sets.

[3] that is, maps of spaces $X \to Y$ inducing isomorphisms of Pontryagin algebras $\mathrm{H}_*(\Omega X; k) \xrightarrow{\sim} \mathrm{H}_*(\Omega Y; k)$.

(b) *For any pointed simplicial set* $X \in \mathtt{sSet}_*$, *there is a natural isomorphism*

$$\mathrm{HR}_*(\Sigma X, \mathcal{H}) \,\cong\, \mathrm{HH}_*(X, \mathcal{H}; k)\ , \tag{1.2.8}$$

where $\mathrm{HH}_*(X, \mathcal{H}; k)$ *is the Pirashvili-Hochschild homology of the commutative algebra* $\mathcal{H}$ *with coefficents in* k *viewed as an* $\mathcal{H}$*-module via the Hopf algebra counit* $\varepsilon : \mathcal{H} \to k$.

The proof of part (a) of Theorem 1.2.6 is based on Milnor's classical FK-construction [114] that gives a simplicial group model for the space $\Omega\Sigma|X|$. Part (b) follows easily from (a).

Theorem 1.2.6 has strong implications: in particular, it allows one to compute the representation homology of suspensions in a completely explicit way. Indeed, it is known that ΣX for any pointed connected space X is *rationally* homotopy equivalent to a bouquet of spheres of dimension ≥ 2. Since representation homology depends only on the rational homotopy type of a space, the isomorphism (1.2.8) together with Pirashvili's computations [133] of higher Hochschild homology of spheres, implies (*cf.* Proposition 5.3.2)

Proposition 1.2.9. *For any pointed connected space* X *of finite type, there is an isomorphism*

$$\mathrm{HR}_*(\Sigma X, G) \,\cong\, \mathrm{Sym}_k\Bigg(\bigoplus_{n\geq 1} \mathrm{H}_n(X; \mathfrak{g}^*)[n]\Bigg)\ ,$$

where $\mathfrak{g}^*$ *is the linear dual of the Lie algebra of* G, *and* $\mathrm{H}_*(X; \mathfrak{g}^*)$ *is the (singular) homology of the space* X *with constant coefficients in* $\mathfrak{g}^*$.

In particular, since $\mathbb{S}^n \cong \Sigma\mathbb{S}^{n-1}$, Proposition 1.2.9 implies

$$\mathrm{HR}_*(\mathbb{S}^n, G) \,\cong\, \mathrm{Sym}_k(\mathfrak{g}^*[n-1])\ ,\ n \geq 2\ . \tag{1.2.10}$$

In Section 6, we look at representation homology of simply-connected spaces. We recall that, by a fundamental theorem of Quillen [136], to any such space X one can associate a connected (chain) DG Lie algebra $\mathcal{L}_X$ called a Quillen model of X. The Lie algebra $\mathcal{L}_X$ encodes the rational homotopy type of X and hence determines any homotopy invariant of X over $\mathbb{Q}$. It is natural to ask how to express the representation homology of X over $\mathbb{Q}$ in terms of $\mathcal{L}_X$. The answer to this question is given by the following result (*cf.* Theorem 6.2.2), which we refer to as the 'Comparison Theorem' and which is technically the main theorem of Part I.

Theorem 1.2.11 (Comparison Theorem). *Let X be a 1-connected pointed topological space of finite rational type. Then, for any affine algebraic group G defined over $\mathbb{Q}$ with Lie algebra $\mathfrak{g}$, there is an isomorphism of graded commutative $\mathbb{Q}$-algebras*

$$\mathrm{HR}_*(X, G) \cong \mathrm{HR}_*(\mathcal{L}_X, \mathfrak{g}) \,. \tag{1.2.12}$$

In Theorem 1.2.11, $\mathrm{HR}_*(\mathcal{L}_X, \mathfrak{g})$ denotes the representation homology of the DG Lie algebra $\mathcal{L}_X$ with coefficients in the (finite-dimensional) Lie algebra $\mathfrak{g}$, as defined in our earlier paper [24][4]. Using the results of [24], we can restate Theorem 1.2.11 in terms of a Sullivan model of X. Recall that a Sullivan model of X is a commutative cochain DG algebra $\mathcal{A}_X$ related to $\mathcal{L}_X$ via a quasi-isomorphism $\mathrm{C}^*(\mathcal{L}_X, \mathbb{Q}) \simeq \mathcal{A}_X$, where $\mathrm{C}^*(\mathcal{L}_X, \mathbb{Q})$ is the Chevaley-Eilenberg cochain complex of $\mathcal{L}_X$. Such a commutative DG algebra model always exists and provides an alternative way to describe the rational homotopy type of X (see [58]). Now, for a fixed $\mathcal{A}_X$, we let $\bar{\mathcal{A}}_X$ denote the augmentation ideal of $\mathcal{A}_X$ corresponding to the basepoint of X, and define $\mathfrak{g}(\bar{\mathcal{A}}_X) := \mathfrak{g} \otimes \bar{\mathcal{A}}_X$ to be the current Lie algebra of $\mathfrak{g}$ over $\bar{\mathcal{A}}_X$. Then, as consequence of [24, Theorem 6.5] and our Comparison Theorem, we get (*cf.* Theorem 6.2.3)

Theorem 1.2.13. (a) *For any 1-connected pointed space X of finite $\mathbb{Q}$-type, there is an*

[4]We will review the definition of representation homology of Lie algebras in Section 6.1.

isomorphism

$$\mathrm{HR}_*(X,G)\;\cong\;\mathrm{H}^{-*}(\mathfrak{g}(\bar{\mathcal{A}}_X);\mathbb{Q})\;, \tag{1.2.14}$$

where the right-hand side is the classical (Chevalley-Eilenberg) cohomology of the DG Lie algebra $\mathfrak{g}(\bar{\mathcal{A}}_X)$ equipped with homological grading.

(b) *If G is a* reductive *affine algebraic group over k, then*

$$\mathrm{HR}_*(X,G)^G\;\cong\;\mathrm{H}^{-*}(\mathfrak{g}(\mathcal{A}_X),\mathfrak{g};\mathbb{Q})\;, \tag{1.2.15}$$

where the right-hand side is the relative Lie algebra cohomology of the pair $\mathfrak{g}\subset\mathfrak{g}(\mathcal{A}_X)$.

Part (b) of Theorem 1.2.13 allows us to compute the G-invariant part of representation homology of some classical simply-connected spaces. For example, we have (*cf.* Theorem 6.4.2 and Corollary 6.4.4)

Theorem 1.2.16. *Let G be a complex reductive Lie group of rank l. Let $m_1,\ldots,m_l$ denote the exponents of the Lie algebra of G. Then, for any $r\geq 1$, there is an isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(\mathbb{CP}^r,G)^G\;\cong\;\mathrm{Sym}_k(\xi_1^{(i)},\xi_3^{(i)},\ldots,\xi_{2r-1}^{(i)}\;:\;i=1,2,\ldots,l)\;, \tag{1.2.17}$$

where each generator $\xi_{2s-1}^{(i)}$ has homological degree $2rm_i+2s-1$, $s=1,2,\ldots,r$.

We deduce the result of Theorem 1.2.16 from the so-called Strong Macdonald Conjecture, a celebrated conjecture in representation theory proposed by Feigin and Hanlon in the early 80s and proved by Fishel, Grojnowski and Teleman in 2008 (see [60]). It is interesting to note that the isomorphism (1.2.17) has a natural topological interpretation: it is given by a canonical algebra map which we call the *Drinfeld homomorphism*:

$$\mathrm{Sym}_k\big[\bigoplus_{i=1}^{l}\mathrm{H}_*^{\mathbb{S}^1,\,(m_i)}(\mathscr{L}X;\,\mathbb{C})\big]\;\to\;\mathrm{HR}_*(X,G)^G\;. \tag{1.2.18}$$

The domain of (1.2.18) is the graded symmetric algebra of the direct sum of certain eigenspaces of the (reduced) $\mathbb{S}^1$-equivariant homology of the free loop space $\mathscr{L}X$ of a simply-connected space X (see Section 6.5). The Drinfeld homomorphism exists for *any* simply-connected space X, and it happens to be an isomorphism in the following cases: $X = \mathbb{S}^n$ $(n \geq 2)$, $X = \mathbb{CP}^r$ $(r \geq 1)$ and $X = \mathbb{S}^m \times \mathbb{CP}^\infty$ (m odd, $m \geq 3$). It would be interesting to find more examples and give an abstract (topological) characterization of all such spaces.

Despite its simple statement, the proof of Theorem 1.2.11 is fairly long and technical: it occupies most of Section 6 and relies heavily on results of Quillen [136]. For reader's convenience, we outline the main steps of the proof in Section 6.3. We close Section 6 with a conjectural generalization of Theorem 1.2.11 to non-simply connected spaces (see Conjecture 6.6.1). Our conjecture is inspired by a recent work of Buijs, Félix, Murillo and Tanré who proposed a natural generalization of Quillen models to non-simply connected spaces (see [27, 28, 29]).

In Section 7, we compute representation homology of some classical non-simply connected spaces. Our examples include closed surfaces of arbitrary genus (both orientable and non-orientable) as well as some three-dimensional spaces (the link complements in $\mathbb{R}^3$ and $\mathbb{S}^3$, the lens spaces $L(p,q)$ and a general closed orientable 3-manifold). The representation homology of surfaces and link complements is expressed in terms of classical Hochschild homology of $\mathcal{O}(G)$ and related commutative algebras. For example, for link complements in $\mathbb{R}^3$, we prove (*cf.* Theorem 7.2.6)

Theorem 1.2.19. *Let L be a link in $\mathbb{R}^3$ obtained as the Alexander closure of a braid $\beta \in B_n$. Then the representation homology of the complement of its (regular) neighborhood* *in $\mathbb{R}^3$ is given by*

$$\mathrm{HR}_*(\mathbb{R}^3 \backslash L, G) \;\cong\; \mathrm{HH}_*(\mathcal{O}(G^n),\; \mathcal{O}(G^n)_\beta)\;. \tag{1.2.20}$$

In (1.2.20), the right-hand side is the usual Hochschild homology of the associative algebra $\mathcal{O}(G^n)$ with bimodule coefficients. The bimodule $\mathcal{O}(G^n)_\beta$ is isomorphic to $\mathcal{O}(G^n) = \mathcal{O}(G)^{\otimes n}$ as a left module, while the right action of $\mathcal{O}(G^n)$ is twisted by an element β viewed as an automorphism of $\mathcal{O}(G)^{\otimes n}$ via the Artin representation of the braid group B_n. The representation homology of the classical lens space $L(p, q)$ and, more generally, of an arbitrary closed orientable 3-manifold M is expressed in terms of its Heegaard splitting by a differential 'Tor' functor; this gives rise to an Eilenberg-Moore spectral sequence converging to $\mathrm{HR}_*(M, G)$ (see Theorem 7.2.24 and Theorem 7.2.26 for precise statements). We should mention that the key to our calculations in Section 7 is Lemma 4.2.8 which says that the derived representation functor (1.1.3) preserves homotopy pushouts. Since the representation functor (1.1.3) is not (left) Quillen (see Remark after Lemma 4.2.4), this result is not immediate and requires a fairly technical proof which we defer to Section 11.

In Section 7, we also discuss a multiplicative version of the derived Harish-Chandra Conjecture proposed in [24]. If G is a reductive group with a maximal torus $T \subset G$ and W is the associated Weyl group, then for any space X, there is a natural restriction map

$$\mathrm{HR}_*(X, G)^G \to \mathrm{HR}_*(X, T)^W\;, \tag{1.2.21}$$

which we call the derived Harish-Chandra homomorphism (*cf.* [24, Section 7]). By the classical Chevalley Restriction Theorem, formula (1.2.10) implies that the map (1.2.21) is an isomorphism for any odd-dimensional sphere $X = \mathbb{S}^{2p+1}$.

We conjecture that (1.2.21) is also an isomorphism for the *product of two odd-dimensional spheres*, $X = \mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ $(p, q \geq 0)$, at least when G is a classical group of type GL_n, SL_n or Sp_{2n} $(n \geq 1)$. In the simply-connected case (for $p, q \geq 1$), this conjecture was stated (in terms of representation homology of Lie algebras) in [24], where some of its consequences and special cases were proved. In the case of the two-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, it can be reformulated in the following explicit form (see Section 7.1, Conjecture 7.1.9):

$$\mathrm{HR}_*(\mathbb{T}^2, G)^G \cong [\mathcal{O}(T \times T) \otimes \Lambda_k^*(\mathfrak{h}^*)]^W ,$$

where $\mathfrak{h}$ is the Lie algebra of T (i.e., a Cartan subalgebra of $\mathfrak{g}$). As for the Drinfeld homomorphism, it would be interesting to find more examples of spaces, for which the derived Harish-Chandra map (1.2.21) is an isomorphism, and give an abstract characterization of all such spaces.

In Chapter 8, we give another interpretation of representation homology as the Hochschild-Mitchell homology of a certain bifunctor on the category of finitely generated free groups $\mathfrak{G}$. Such an interpretation is useful for several reasons. First, it allows us to define representation *cohomology* in a natural way (by simply replacing the Hochschild-Mitchell homology with the Hochschild-Mitchell cohomology of the same bifunctor). Second, it suggests that it is natural to extend the definition of representation (co)homology by taking the Hochschild-Mitchell (co)homology of $\mathfrak{G}$ with coefficients in an arbitrary bifunctor D: i.e., $\mathrm{HR}(D) := \mathrm{HH}(\mathfrak{G}, D)$. Third and most important, it exhibits a close analogy with *topological* Hochschild homology, which is known to be isomorphic to the Hochschild-Mitchell homology of the category $\mathfrak{G}_{\mathrm{ab}}$ of finitely generated free *abelian* groups (see [132]). Motivated by this analogy, we construct functorial trace maps

$$\mathrm{DTr}_n^{\mathfrak{G}}(D): \mathrm{H}_*(\mathrm{Aut}(\mathbb{F}_n), D_n) \to \mathrm{HR}_*(D), \quad \forall\, n \geq 1,$$

relating homology of the automorphism groups of f.g. free groups with appropriate coefficients to representation homology. These maps are compatible with natural inclusions $\mathrm{Aut}(\mathbb{F}_n) \hookrightarrow \mathrm{Aut}(\mathbb{F}_{n+1})$ and hence have an stable limit as $n \to \infty$. The corresponding stable map $\mathrm{DTr}_\infty^{\mathfrak{G}}(D): \mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty) \to \mathrm{HR}_*(D)$ can be viewed a non-abelian analogue of the classical Dennis trace relating topological Hochschild homology to stable homology of general linear groups. We conjecture that the map $\mathrm{DTr}_\infty^{\mathfrak{G}}(D)$ is actually an isomorphism, whenever D is a *polynomial* bifunctor (*cf.* Conjecture 8.2.6). This is a non-abelian analogue of a theorem of Scorichenko [146].

1.3 Compendium

Part II of this thesis form a compendium to this part. In Section 10, we describe an abstract monoidal version of the classical Dold-Kan correspondence relating the category of (non-negatively graded) DG $\mathcal{P}$-algebras and the category of simplicial $\mathcal{P}$-algebras for an arbitrary k-linear operad $\mathcal{P}$. This monoidal version of the Dold-Kan correspondence is needed for our proof of Comparison Theorem in Section 6. The main result of Section 10 is Theorem 10.3.2, which states that when k is a field of characteristic 0, there is a Quillen equivalence between the category of (non-negatively graded) DG $\mathcal{P}$-algebras and the category of simplicial $\mathcal{P}$-algebras. Various special cases of this theorem have appeared in the literature. First of all, when $\mathcal{P}$ is the Lie operad, a slightly weaker version (namely, a Quillen equivalence between the category of *positively graded* DG Lie algebras and reduced simplicial Lie algebras) was proved in [136, Part I, Theorem 4.6]. In *loc. cit.* Quillen also outlines a proof for the commutative operad (which controls commutative unital k-algebras) under the same reducedness assumptions. For general (nonreduced) commutative algebras, the proof of the Dold-Kan correspondence is given in [157, Proposition A.1]. The case of the associative operad is treated in greater generality (for any commutative ring k) in [145], where the DG associative algebras and simplicial associative algebras are viewed as monoids in the (symmetric) monoidal model categories of chain complexes and simplicial k-modules, respectively. In this case, the Dold-Kan correspondence follows from an abstract comparison theorem between monoids in different (symmetric) monoidal model categories. The arguments that establish each of these special cases seem to apply only to the case in hand. To the best of our knowledge, a unified proof for any linear operad is missing in the literature. Our Theorem 10.3.2 fills in this gap. The key argument we use to prove Theorem 10.3.2 is sketched in [63, Remark 6.4.5] in the special case of the commutative operad. Theorem 10.3.2 is crucial for the proof of our Comparison Theorem (Theorem 1.2.11). While Quillen's original result for reduced DG Lie algebras is sufficient for this proof, the full strength of Theorem 10.3.2 is needed to prove Proposition 6.3.10, which is an interesting result on its own.

In Section 11 of Part II, we collect some technical facts about simplicial commutative algebras. Most of these facts should be well known to experts and go back to the classical works of André [6] and Quillen [137]. However, the main results of this section – Proposition 11.3.13 and Proposition 11.3.14 – did not seem to appear in the literature in this form and generality. These propositions are needed to prove Lemma 4.2.8, which we use in the proofs of our main theorems in Section 4 and Section 5 and also for explicit computations in Section 7.

While a weaker version of Lemma 4.2.8 (that can be proven directly) suffices for the results in Section 4 and Section 5, the full strength of this Lemma is required for computations in Section 7.

1.4 Outline of this Part

Part I of this thesis is organized as follows. In Chapter 2, we introduce notation and recall some basic facts about simplicial sets and spaces. In Chapter 3, we review Pirashvilli's construction of higher Hochschild homology. In Chapter 4, we define representation homology in two equivalent ways (via the derived representation functor and the Pirashvilli-type construction) and prove some basic properties. In Section 4, we establish the isomorphism between the representation homology of suspensions and higher Hochschild homology. In Chapter 6, we prove our main Comparison Theorem for simply-connected spaces and discuss its implications. In Chapter 7, we explicitly compute representation homology for certain non-simply connected spaces. Finally, in Chapter 8, we identify representation homology in terms of Hochschild-Mitchell homology and construct a non-abelian analogue of the Dennis trace map relating representation homology to the stable homology of automorphism groups of finitely generated free groups.

CHAPTER 2

PRELIMINARIES

In this section, we introduce notation and recall some basic facts about simplicial sets. Standard references for this material are [112], [67] and [164, Chapter 8].

2.1 Simplicial objects

Let Δ denote the simplicial category. Recall that the objects of Δ are the finite ordered sets $[n] := \{0, 1, \ldots, n\}$, $n \geq 0$ and the morphisms are the (weakly) order preserving maps $[n] \to [m]$. A *simplicial object* in a category $\mathscr{C}$ is a contravariant functor from Δ to $\mathscr{C}$: *i.e.*, $\Delta^{\mathrm{op}} \to \mathscr{C}$. The simplicial objects in $\mathscr{C}$ form a category, the morphisms in which are given by natural transformations of functors. We denote this category by $s\mathscr{C}$. If $X \in \mathrm{Ob}(s\mathscr{C})$, we write X_n for $X([n])$.

The category Δ is generated by two distinguished classes of morphisms $\{\delta^i\}_{0\leq i\leq n}^{n\geq 1}$ and $\{\sigma^j\}_{0\leq j\leq n}^{n\geq 0}$, whose images under $X \in s\mathscr{C}$ are called the *face maps* and *degeneracy maps* of X, respectively. The map $\delta^i : [n-1] \to [n]$ is the (unique) injection that does not contain "i" in its image; the corresponding face map is denoted by $d_i := X(\delta^i) : X_n \to X_{n-1}$. Similarly, for $n \geq 0$, the map $\sigma^i : [n+1] \to [n]$ is the (unique) surjection in Δ that takes value "i" twice. The image of σ^i under X is the degeneracy map $s_i := X(\sigma^i) : X_n \to X_{n+1}$. The face and degeneracy maps of a simplicial object satisfy the following *simplicial* *relations*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if } i < j \\ d_i s_j &= s_{j-1} d_i \quad \text{if } i < j \\ d_i s_j &= s_j d_{i-1} \quad \text{if } i > j+1 \\ s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j \\ d_i s_j &= \mathrm{id} \qquad\ \ \text{if } i = j\,,\ j+1\,. \end{aligned} \tag{2.1.1}$$

Thus, a simplicial object in $s\mathscr{C}$ is determined by a family $X = \{X_n\}_{n\geq 0}$ of objects in $\mathscr{C}$ together with morphisms $d_i : X_n \to X_{n-1}$ and $s_j : X_n \to X_{n+1}$ satisfying the relations (2.1.1). The object X_0 is usually called the "set" of n-*simplices* of X, and the 0-simplices are usually called the *vertices* of X.

We let $\mathtt{sSet}$ denote the category of simplicial sets (*i.e.*, simplicial objects in the category $\mathtt{Set}$). A simplicial set X is called *reduced* if the set X_0 of its 0-simplices is a singleton. The full subcategory of $\mathtt{sSet}$ whose objects are reduced simplicial sets will be denoted $\mathtt{sSet}_0$. A simplicial set X is called *pointed* if there are distinguished simplices $x_n \in X_n$, one in each degree, such that $x_n = s_0(x_{n-1})$ for all $n \geq 1$. The sequence $(x_0, x_1, x_2, \ldots) \in \prod_{n\geq 0} X_n$ is called a basepoint of X. The category of pointed simplicial sets will be denoted $\mathtt{sSet}_*$. Note that $\mathtt{sSet}_0$ can also be viewed as a full subcategory of $\mathtt{sSet}_*$ as every reduced simplcial set has a canonical (unique) basepoint.

Given $X \in \mathtt{sSet}$, the set of *nondegenerate* n-simplices of X is defined to be

$$\overline{X}_n := X_n \setminus \bigcup_{i=0}^{n-1} s_i(X_{n-1})\,.$$

Every element of X_n can be uniquely expressed in terms of the nondegenerate elements of X (see [70, Lemma 11] for a precise statement). In particular, a simplicial set can be defined by specifiying its nondegenerate simplices together with the restriction of each face map to the set of nondegenerate simplices.

We give a few basic examples of simplicial sets that will be used in this thesis.

Discrete simplicial objects

To any object $A \in \mathscr{C}$ one can associate a simplicial object $A_* \in s\mathscr{C}$, with $A_n = A$ and d_i, s_j being the identity map of A for all n, i, j. This gives a fully faithful embedding $\mathscr{C} \hookrightarrow s\mathscr{C}$. The objects of $s\mathscr{C}$ arising this way are called *discrete simplicial objects*.

Geometric simplices

The n-dimensional geometric simplex is the topological space

$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1\,,\, x_i \geq 0\,\}\,.$$

Let e_i denote the vertex of Δ^n with i-th coordinate 1. For any morphism $f : [m] \to [n]$ in Δ, there is a (unique) linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ sending e_i to $e_{f(i)}$, that restricts to a map of topological spaces $f^* : \Delta^m \to \Delta^n$. The collection $\Delta^* := \{\Delta^n\}_{n\geq 0}$ forms a *cosimplicial space*, *i.e.*, a (covariant) functor $\Delta \to \mathtt{Top}$, where $\mathtt{Top}$ denotes the category of (compactly generated weakly Hausdorff) topological spaces. This functor is faithful: it gives a topological realization of the simplicial category, which was historically the first definition of Δ.

Standard simplices

Let $Y : \Delta \hookrightarrow \mathtt{sSet}$ denote the Yoneda embedding. The functor Y assigns to $[n]$ a simplicial set $\Delta[n]_*$ called the *standard n-simplex*. Explicitly, $\Delta[n]_*$ is given by

$$\Delta[n]_k \;:=\; \mathrm{Hom}_\Delta([k],[n]) \;\cong\; \{(n_0,\ldots,n_k)\,|\, 0 \le n_0 \le \ldots \le n_k \le n\}\,,$$

where a function $f : [k] \to [n]$ is identified with the sequence of its values $(f(0),\ldots,f(k))$. Under this identification, the nondegenerate simplices correspond to *strictly* increasing functions, and the face and degeneracy maps in $\Delta[n]_*$ are given by

$$d_i(n_0,\ldots,n_k) \;=\; (n_0,\ldots,\hat{n}_i,\ldots,n_k)\,,\quad s_j(n_0,\ldots,n_k) \;=\; (n_0,\ldots,n_j,n_j,\ldots,n_k)\,.$$

By Yoneda Lemma, for any simplicial set X, there is a natural bijection

$$\mathrm{Hom}_{\mathtt{sSet}}(\Delta[n]_*, X) \;\cong\; X_n\,,$$

which shows that $\Delta[n]_*$ (co)represents the functor: $\mathtt{sSet} \to \mathtt{Set}\,,\ X \mapsto X_n$.

Simplicial spheres

The Yoneda functor $Y : \Delta \to \mathtt{sSet}$ can be also regarded as a cosimplicial object in the category of simplicial sets. In particular, for any $n \ge 1$, there are $n+1$ coface maps $d^i : \Delta[n-1]_* \to \Delta[n]_*$, $0 \le i \le n$. Using these maps, we define the *boundary* of $\Delta[n]_*$ to be the simplicial subset

$$\partial\Delta[n]_* \;:=\; \bigcup_{0\le i\le n} d^i(\Delta[n-1]_*) \;\subset\; \Delta[n]_*\,,$$

The *simplicial n-sphere* is then defined to be the corresponding quotient set $\mathbb{S}^n_* := \Delta[n]_*/\partial\Delta[n]_*$. It is easy to see that the only nondegenerate simplices in $\mathbb{S}^n_*$ are in degree 0 and n, with $\overline{\mathbb{S}}^n_0 = \{*\}$ and $\overline{\mathbb{S}}^n_n = \{S\}$, where S is the image of the map $\mathrm{id} \in \Delta[n]_n$ in $\mathbb{S}^n_n$. Note that $d_i(S) = s_0^{n-1}(*)$ for all i. Thus, the simplicial structure of $\mathbb{S}^n_*$ reflects the standard CW decomposition of the n-sphere $\mathbb{S}^n$ with one cell in dimension 0 and one cell in dimension n.

The simplicial 1-sphere $\mathbb{S}^1_*$ is called the *simplicial circle*. By Example 2.1, we have $\Delta[1]_k \cong \{(\underbrace{0,\dots,0}_{i},\underbrace{1,\dots,1}_{k+1-i}) \,|\, i = 0,1,\dots,k+1\}$ and $\partial\Delta[0]_k = \{(0,\dots 0),(1,\dots,1)\}$. Hence, $\mathbb{S}^1_*$ is given explicitly by

$$\mathbb{S}^1_k \cong \{(\underbrace{0,\dots,0}_{i},\underbrace{1,\dots,1}_{k+1-i}) \,|\, i = 1,\dots,k+1\}\,,$$

with $(0,\dots,0)$ corresponding to the basepoint $*$.

2.2 Geometric realization

There is an important functor $|-| : \mathtt{sSet} \to \mathtt{Top}$ assigning to each simplicial set X a topological space $|X|$ called the *geometric realization* of X. Explicitly, the space $|X|$ is defined by

$$|X| := \bigsqcup_{n\geq 0}(X_n \times \Delta^n)/\sim\,,$$

where each set X_n is equipped with discrete topology and the equivalence relation is given by

$$(d_i x, p) \sim (x, d^i p) \text{ for } (x,p) \in X_n \times \Delta^{n-1}$$
$$(s_j x, p) \sim (x, s^j p) \text{ for } (x,p) \in X_{n-1} \times \Delta^n\,.$$

More formally (see, e.g., [140, Section 1.3]), the functor $|-| : \mathtt{sSet} \to \mathtt{Top}$ can be defined as the (left) Kan extension $|-| = \mathrm{Lan}_Y(\Delta^*)$ of the geometric simplex Δ^*

along the Yoneda embedding $Y : \Delta \to \mathtt{sSet}$. It follows from this definition that $|\Delta[n]_*| \cong \Delta^n$ for all $n \geq 0$, and in general, $|X| \cong \operatorname{colim} \Delta^n$, where the colimit is taken over all morphisms of $\Delta[n]_* \to X$, $n \geq 0$.

2.3 Simplicial sets and spaces

The category $\mathtt{sSet}$ has a standard model structure, where the weak equivalences are the morphisms inducing weak homotopy equivalences of the corresponding geometric realizations. The cofibrations are levelwise injective maps and the fibrations are the Kan fibrations (see [112, §7]). This structure gives a model structure on $\mathtt{sSet}_0$.

Let $(X, *)$ be a pointed topological space. The *(total) singular complex* of X is a simplicial set $\mathrm{S}_*(X)$ defined by $\mathrm{S}_n(X) := \mathrm{Hom}_{\mathtt{Top}}(\Delta^n, X)$. The *Eilenberg subcomplex* $\mathrm{ES}_*(X)$ of $\mathrm{S}_*(X)$ is

$$\mathrm{ES}_n(X) := \{f : \boldsymbol{\Delta}^n \to X : f(v_i) = * \text{ for all vertices } v_i \in \boldsymbol{\Delta}^n\}.$$

If X is connected, the natural inclusion $\mathrm{ES}_*(X) \hookrightarrow \mathrm{S}_*(X)$ is a weak equivalence of simplicial sets. Further, if we restrict ES to the category $\mathtt{Top}_{0,*}$ of connected pointed spaces, we get the pair of adjoint functors

$$|-| : \mathtt{sSet}_0 \rightleftarrows \mathtt{Top}_{0,*} : \mathrm{ES}\,,$$

which induce mutually inverse equivalences of the homotopy categoriess: $\mathrm{Ho}(\mathtt{sSet}_0) \simeq \mathrm{Ho}(\mathtt{Top}_{0,*})$. This equivalence justifies the following standard convention which we will follow throughout the thesis.

Convention. We shall not notationally distinguish between a reduced simplicial set X and its geometric realization $|X|$. Nor shall we distinguish notationally between a topological space and a (reduced) simplicial model of that space.

CHAPTER 3

HIGHER HOCHSCHILD HOMOLOGY

Recall that for an arbitrary associative k-algebra A and an A-bimodule M, the classical Hochschild chain complex $\mathrm{C}_*(A, M)$ is defined by $\mathrm{C}_n(A, M) := M \otimes A^{\otimes n}$, $n \geq 0$, with differential $d : \mathrm{C}_n(A, M) \to \mathrm{C}_{n-1}(A, M)$ given by the formula

$$d(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} \,,$$

where $a_0 \in M$ and $a_1, a_2, \ldots, a_n \in A$. The homology of $\mathrm{C}_*(A, M)$ is denoted $\mathrm{HH}_*(A, M)$ and called the *Hochschild homology of A with coefficients in M*.

For *commutative* algebras, there is a remarkable generalization of the Hochschild complex, due to Pirashvili [133], that depends on an arbitrary simplicial set X and specializes for $X = \mathbb{S}^1_*$ to the original classical construction. In this section, we review the definition of Pirashvili's complex and give two natural interpretations of its homology groups in terms of Loday construction and functor tensor products. All results in this section can be found in (or easily derived from) Pirashvili's papers [133, 134].

3.1 The Pirashvili-Hochschild complex

Let $\mathfrak{F}_*$ denote the category of finite pointed sets with objects $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$, and morphisms $f : [n] \to [m]$ being arbitrary set maps such that $f(0) = 0$. Let $F : \mathfrak{F}_* \to \mathtt{Vect}_k$ be a covariant functor. We extend F to the category $\mathtt{Set}_*$ of *all* pointed sets in a natural way, using the left Kan extension along the inclusion $\mathfrak{F}_* \hookrightarrow \mathtt{Set}_*$. We keep the notation F for the extended functor: explicitly, F :

$\mathtt{Set}_* \to \mathtt{Vect}_k$ is given by $F(X) = \operatorname{colim} F([n])$, where the colimit is taken over all pointed inclusions $[n] \hookrightarrow X$.

Given a pointed simplicial set $X \in \mathtt{sSet}_*$, we define a simplicial k-vector space $F(X)$ as the composition of functors

$$F(X):\ \Delta^{\mathrm{op}} \xrightarrow{X} \mathtt{Set}_* \xrightarrow{F} \mathtt{Vect}_k\ . \tag{3.1.1}$$

We denote the homotopy groups of $F(X)$ by $\pi_* F(X)$ and recall that $\pi_* F(X) := \mathrm{H}_*[N(F(X))]$, where N is the Dold-Kan normalization functor (see Section 10.1).

Now, any commutative k-algebra A and an A-module M (viewed as a symmetric bimodule) give rise to a functor $\mathfrak{F}_* \to \mathtt{Vect}_k$ that assigns to the set $[n]$ the vector space $M \otimes A^{\otimes n}$ and to a pointed map $f : [n] \to [m]$, the action of f on $M \otimes A^{\otimes n}$ given by

$$f_*(a_0 \otimes a_1 \otimes \ldots \otimes a_n) := b_0 \otimes b_1 \otimes \ldots b_m\ ,$$

where $b_j := \prod_{i \in f^{-1}(j)} a_i$ for $j = 0, 1, \ldots m$. Following [133], we denote this functor by $\mathcal{L}(A, M)$, and for a pointed simplicial set $X \in \mathtt{sSet}_*$, define

$$\mathrm{HH}_*(X, A, M) := \pi_* \mathcal{L}(A, M)(X)\,.$$

Thus, $\mathrm{HH}_*(X, A, M)$ is the homology of the complex $\mathrm{C}_*(X, A, M) := N[\mathcal{L}(A, M)(X)]$, which we call the *Pirashvili-Hochschild complex* of A with coefficients in M associated to X.

Example 3.1.2. Let $X = \mathbb{S}^1_*$ be the simplicial circle. Recall that the set of n-simplices $\mathbb{S}^1_n$ can be identified with the set of monotone sequences of 0's and 1's of length $n+1$ modulo the identification $(0, 0, \ldots, 0) \sim (1, 1, \ldots, 1)$ (see Section 2.1). For a nonzero sequence $x \in \mathbb{S}^1_n$, let $n(x)$ denote the position of the first 1. The map $x \mapsto n(x) - 1$ identifies $\mathbb{S}^1_n$ with $[n]$. Under this identification, the degeneracy map $s_i : [n] \to [n+1]$ corresponds to the unique monotone injection skiping $i+1$ in its image and the face map $d_i : [n] \to [n-1]$ is given by $d_i(j) = j$ for $j < i$, $d_i(i) = i$ for $i < n$, $d_n(n) = 0$ and $d_i(j) = j-1$ for $j > i$. From this description of $\mathbb{S}^1_*$, it is easy to see that the Pirashvili complex $\mathrm{C}_*(\mathbb{S}^1, A, M)$ for $\mathbb{S}^1$ is precisely the classical Hochschild complex $\mathrm{C}_*(A, M)$. Thus, $\mathrm{HH}_*(\mathbb{S}^1, A, M) = \mathrm{HH}_*(A, M)$ for any commutative algebra A and A-module M. In a similar way, one can exlicitly describe the Pirashvili complex $\mathrm{C}_*(\mathbb{S}^n, A, M)$ for the n-dimensional simplicial sphere $\mathbb{S}^n_*$. The corresponding homology groups $\mathrm{HH}_*(\mathbb{S}^n, A, M)$ are denoted $\mathrm{HH}^{[n]}_*(A, M)$ and called in [133] *the Hochschild homology of* (A, M) *of order* n.

In Part I, we will mostly deal with two cases: $M = A$ and $M = k$, where in the last case the module structure on k comes from an augmentation $A \to k$. To simplify the notation we will write $\mathrm{HH}_*(X, A)$ for $\mathrm{HH}_*(X, A, A)$ and regard $X \mapsto \mathrm{HH}_*(X, A)$ as a functor on the category of (pointed) simplicial sets assuming A to be fixed. We will refer to this functor as a *higher Hochschild homology* of spaces.

3.2 Loday construction

We now give a useful interpretation of higher Hochschild homology that exhibits the functor $\mathcal{L}$ as a special case of the Loday construction [105, 106]. The starting point is the simple observation that the category $\mathtt{CommAlg}_k$ of commutative algebras is a cocartesian monoidal category tensored over the category $\mathtt{Set}$. This means that $\mathtt{Set}$ acts on $\mathtt{CommAlg}_k$ in the natural way, *i.e.*, there is a bifunctor

$$\mathtt{Set} \times \mathtt{CommAlg}_k \to \mathtt{CommAlg}_k\,, \quad (S, A) \mapsto S \otimes A\,, \tag{3.2.1}$$

where $S \otimes A := \otimes_{s \in S} A$ is defined to be the S-indexed coproduct of copies of A in the category $\mathtt{CommAlg}_k$. Now, given a simplicial set X, we may consider the composition of functors

$$X \otimes A : \ \Delta^{\mathrm{op}} \xrightarrow{X} \mathtt{Set} \xrightarrow{-\otimes A} \mathtt{CommAlg}_k \to \mathtt{Vect}_k \ ,$$

which, abusing notation, we keep denoting $X \otimes A$.

Lemma 3.2.2. *There is a natural isomorphism of functors* $X \otimes A \ \cong \ \mathcal{L}(A, A)(X)$.

Proof. This is immediate from the following commuative diagram

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, "X"] \arrow[dr, "X"'] & \mathtt{Set} \arrow[r, "(-\otimes A)"'] & \mathtt{Vect}_k \\ & \mathtt{Set}_* \arrow[ur, "\mathcal{L}(A,A)"'] & \end{tikzcd}$$

where the vertical arrow is the forgetful functor from $\mathtt{sSet}_*$ to $\mathtt{sSet}$. $\square$

It follows from Lemma 3.2.2 that $\mathrm{HH}_*(X, A) \cong \pi_*(X \otimes A)$. As a consequence, we have

Proposition 3.2.3. *For any* $X \in \mathtt{sSet}_*$ *and* $A \in \mathtt{CommAlg}_k$, $\mathrm{HH}_*(X, A)$ *has the structure of a graded commutative* k*-algebra which is independent of the choice of a basepoint in* X.

Proof. Indeed, by definition, $X \otimes A$ has the structure of a simplicial commutative k-algebra that depends only on the underlying simplicial set of X. $\square$

3.3 Hochschild homology as a derived functor

There is another, more conceptual way to define higher Hochschild homology, using homological algebra of functor categories over PROPs. Recall that a PROP is a permutative category $(\mathcal{P}, \boxtimes)$ whose set of objects is indexed by (or identified with) the natural numbers $\mathbb{N}$ and whose monoidal structure $\boxtimes$ is given by addition in $\mathbb{N}$ (see [117]). A k-algebra over a PROP $\mathcal{P}$ is a strict symmetric monoidal functor from $\mathcal{P}$ to the tensor category $\mathtt{Vect}_k$.

To define Hochschild homology we take $\mathcal{P}$ to be a category $\mathfrak{F}$ of finite sets with monoidal structure given by disjoint union. More precisely, we let $\mathfrak{F}$ denote the full subcategory of $\mathtt{Set}$ whose objects are the sets $\underline{n} := \{1, 2, \ldots, n\}$ for $n \geq 0$ (where, by convention, $\underline{0} = \varnothing$) and morphisms are arbitrary set maps. The monoidal structure on $\mathfrak{F}$ is given by $\underline{n} \boxtimes \underline{m} = \underline{n+m}$. It is well known and easy to prove (see, e.g., [134, Section 2]) that the category of k-algebras over $\mathfrak{F}$ is equivalent to the category $\mathtt{CommAlg}_k$, the equivalence being given by the functor $A \mapsto [(-\otimes A) : \underline{n} \mapsto A^{\otimes n}]$. We will write $\underline{A}$ for the algebra over $\mathfrak{F}$ corresponding to the commutative algebra $A \in \mathtt{CommAlg}_k$.

Now, let $\mathfrak{F}$-$\mathtt{Mod}$ (resp., $\mathtt{Mod}$-$\mathfrak{F}$) denote the category of all covariant (resp., contravariant) functors from $\mathfrak{F}$ to the category of vector spaces. The notation suggests that one should think of the objects of $\mathfrak{F}$-$\mathtt{Mod}$ and $\mathtt{Mod}$-$\mathfrak{F}$ as left and right $\mathfrak{F}$-modules, respectively. These categories are both abelian with enough projective and injective objects. Furthermore, they are related by a bifunctor

$$- \otimes_{\mathfrak{F}} - \,:\, \mathtt{Mod}\text{-}\mathfrak{F} \times \mathfrak{F}\text{-}\mathtt{Mod} \,\to\, \mathtt{Vect}_k$$

that is right exact with respect to each argument, preserves sums and is left balanced (see, e.g., [133, Sect. 1.5]). Explicitly, for a right $\mathfrak{F}$-module $\mathcal{N}$ and a left $\mathfrak{F}$-module $\mathcal{M}$,

$$\mathcal{N} \otimes_{\mathfrak{F}} \mathcal{M} \,=\, \Big[\bigoplus_{n \geq 0} \mathcal{N}(\underline{n}) \otimes_k \mathcal{M}(\underline{n})\Big]/R\,, \tag{3.3.1}$$

where R is the subspace spanned by the vectors of the form $\mathcal{N}(f)x \otimes y - x \otimes \mathcal{M}(f)y$ with $x \in \mathcal{N}(\underline{n})$ and $y \in \mathcal{M}(\underline{m})$ and f running over all maps in

$\mathrm{Hom}_{\mathtt{Set}}(\underline{m}, \underline{n})$.

Next, we consider the functor

$$h : \mathfrak{F} \to \mathtt{Mod}\text{-}\mathfrak{F}\,, \quad \underline{n} \mapsto k[\mathrm{Hom}_{\mathfrak{F}}(-, \underline{n})]\,, \tag{3.3.2}$$

where $k[S]$ denotes the vector space generated by a set S, and extend (3.3.2) to the category simplicial sets in two steps. First, we define a functor $\mathtt{Set} \to \mathtt{Mod}\text{-}\mathfrak{F}$ by taking the left Kan extension of (3.3.2) along the natural inclusion $\mathfrak{F} \hookrightarrow \mathtt{Set}$, and then we extend this degreewise to simplicial sets. Abusing notation, we will continue to denote the resulting functor by $h : \mathtt{sSet} \to \mathtt{sMod}\text{-}\mathfrak{F}$. Composing h with the normalization functor $\underline{N} : \mathtt{sMod}\text{-}\mathfrak{F} \to \mathtt{Ch}_{\geq 0}(\mathtt{Mod}\text{-}\mathfrak{F})$ assigns to every simplicial set X a chain complex and hence an object in the derived category $\mathscr{D}(\mathtt{Mod}\text{-}\mathfrak{F})$ that we denote by $\underline{N}(h(X))$.

Now, recall that any commutative algebra A defines an algebra $\underline{A}$ over the PROP $\mathfrak{F}$ that can be viewed as an object in $\mathfrak{F}\text{-}\mathtt{Mod}$. With this interpretation of A, we have the following result.

Theorem 3.3.3. *For any $X \in \mathtt{sSet}$ and $A \in \mathtt{CommAlg}_k$, there is a natural isomorphism*

$$\mathrm{HH}_*(X, A) \cong \mathrm{H}_*[\underline{N}(h(X)) \otimes^{\boldsymbol{L}}_{\mathfrak{F}} \underline{A}\,]$$

Although Theorem 3.3.3 is not explicitly stated in [133], it can be deduced from results of the present thesis. We do not give a proof of Theorem 3.3.3 here as in the next section, we prove the analogous theorem for representation homology (see Theorem 4.2.15).

CHAPTER 4

REPRESENTATION HOMOLOGY

In this section, we define representation homology of a (reduced) simplicial set in a way parallel to Pirashvili's construction of Hochschild homology. A new key ingredient is Kan's construction [89] of a simplicial group model of the loop space ΩX of a topological space X. We begin by reviewing this classical construction (for details and proofs we refer to [112, Chapter VI] and [67, Chapter V]).

4.1 Kan's loop group construction

Let sGr denote the category of simplicial groups. It has a standard model structure, where the weak equivalences and fibrations of simplicial groups are the weak equivalences and fibrations of the underlying simplicial sets. We note that, unlike sSet, the model category sGr is fibrant: by a classical theorem of Moore, every simplicial group is a Kan complex (see [112, Theorem 17.1]).

Definition 4.1.1. A simplicial group $\Gamma = \{\Gamma_n\}_{n\geq 0}$ is called *semi-free* if there is a sequence of subsets $B_n \subset \Gamma_n$, one in each degree, such that Γ_n is freely generated by B_n, and the set $B = \bigcup_{n\geq 0} B_n$ is closed under degeneracies of Γ, *i.e.*, $s_j(B_{n-1}) \subseteq B_n$ for all $0 \leq j \leq n-1$ and $n \geq 1$. The subset $\overline{B}_n := B_n \backslash \bigcup_{i=0}^{n-1} s_i(B_{n-1})$ is called the set of *nondegenerate generators* of Γ of degree n.

One can show that every element in $\overline{B}_n$ is nondegenerate (when considered as an element of the underlying simplicial set), and a semi-free simplicial group is determined by specifying the sets of nondegenerate generators $\overline{B}_n$ and the face elements of these generators.

Semi-free simplicial groups are cofibrant objects in the model category $\mathtt{sGr}$. The Kan loop group construction provides an important class of semi-free simplicial groups that arise naturally from reduced simplicial sets. To be precise, the Kan construction defines a pair of adjoint functors:

$$\mathbb{G}:\ \mathtt{sSet}_0 \rightleftarrows \mathtt{sGr}\ :\overline{W} \tag{4.1.2}$$

where $\mathbb{G}$ is called the *Kan loop group functor* (or *the cobar construction*) and $\overline{W}$ is the *classifying simplicial complex* (or *the bar construction*). These functors form a Quillen pair, which is actually a Quillen equivalence: i.e., the adjunction (4.1.2) induces an equivalence between the homotopy categories of $\mathtt{sSet}_0$ and $\mathtt{sGr}$ (see [67, Proposition V.6.3]). Combining this with the classical Quillen equivalence between topological spaces and simplicial sets (see Section 2.3):

$$\mathtt{Top}_{0,*} \xrightarrow{\mathrm{ES}} \mathtt{sSet}_0 \xrightarrow{\mathbb{G}} \mathtt{sGr}$$

we get an equivalence of the homotopy categories $\mathrm{Ho}(\mathtt{Top}_{0,*}) \cong \mathrm{Ho}(\mathtt{sSet}_0) \cong \mathrm{Ho}(\mathtt{sGr})$.

For further use, we recall the explicit construction of the functor $\mathbb{G}$. Given a reduced simplicial set $X = \{X_n\}_{n\geq 0}$, the set of n-simplices of $\mathbb{G}X$ is defined by

$$\mathbb{G}X_n \ =\ \langle X_{n+1}\rangle / \langle s_0(x) = 1\,,\ \forall\, x\ \in\ X_n\rangle\ \cong\ \langle B_n\rangle\ ,$$

where $B_n := X_{n+1} \setminus s_0(X_n)$ and the isomorphism is induced by the inclusion $B_n \hookrightarrow X_{n+1}$. The degeneracy maps $s_j^{\mathbb{G}X} : \mathbb{G}X_n \to \mathbb{G}X_{n+1}$ are induced by the degeneracy maps $s_{j+1} : X_{n+1} \to X_{n+2}$ of the simplicial set X, and the face maps $d_i^{\mathbb{G}X} : \mathbb{G}X_n \to \mathbb{G}X_{n-1}$ are given by

$$d_0^{\mathbb{G}X}(x) := (d_1 x)\cdot(d_0 x)^{-1} \quad \text{and} \quad d_i^{\mathbb{G}X}(x) := d_{i+1}(x)\,,\ \forall\, i > 0\,.$$

Conversely, given a simplicial group $\Gamma = \{\Gamma_n\}_{n \geq 0}$, the simplicial set $\overline{W}\Gamma$ is defined by $\overline{W}\Gamma_0 := \{*\}$ and $\overline{W}\Gamma_n := \Gamma_{n-1} \times \Gamma_{n-2} \times \ldots \times \Gamma_0$ for $n \geq 0$. The degeneracy and face maps of $\overline{W}\Gamma$ are given explicitly in [112, §21]. We note that when restricted to *discrete* simplicial groups, the functor $\overline{W}$ coincides with the usual nerve construction, *i.e.*, $\overline{W}\Gamma = \mathrm{B}\Gamma$ for any discrete group Γ.

Proposition 4.1.3. *The Kan loop group $\mathbb{G}X$ of any reduced simplicial set X is semi-free. More precisely, for each $n > 0$, the composite map $\tau : X_n \to \langle X_n \rangle \twoheadrightarrow \mathbb{G}X_{n-1}$ is injective when restricted to the subset $\overline{X}_n \subset X_n$, and the image $\tau(\overline{X}_n) \subset \mathbb{G}X_{n-1}$ forms the set of nondegenerate generators $\overline{B}_{n-1} = \tau(\overline{X}_n)$ in degree $(n-1)$ of the semi-free basis $\{B_n\}_{n \geq 0}$ of $\mathbb{G}X$.*

The following fundamental theorem clarifies the meaning of the Kan loop group construction.

Theorem 4.1.4 (Kan)**.** *For any reduced simplicial set X, there is a weak homotopy equivalence*

$$|\mathbb{G}X| \simeq \Omega|X|\,,$$

where $\Omega|X|$ is the (Moore) based loop space of $|X|$.

A detailed proof of Theorem 4.1.4 can be found in [112] (see *loc. cit.*, Theorem 26.6). Its significance becomes clear from the following considerations. Given any path-connected CW complex Y one can choose a pointed connected simplicial set X' such that $|X'| \simeq Y$. If X is the path-connected component of X' containing the basepoint, then X is a reduced simplicial set such that $|X| \simeq |X'| \simeq Y$ because Y is connected. Hence, applying the Kan loop group construction to X, we get $|\mathbb{G}X| \simeq \Omega Y$. Thus, $\mathbb{G}X$ is a semi-free simplicial group model of the based loop space of Y. In this way, the based loop space of any path-connected CW complex admits a simplicial group model.

4.2 Definition of representation homology

We now introduce several (equivalent) definitions of the representation homology of a reduced simplicial set with coefficients in an algebraic group G. We begin with the construction of derived representation schemes of simplicial groups, following the approach of [23, 21] (see Part II for details).

Derived representation schemes of simplicial groups

Let G be an affine algebraic group scheme over k. By definition, G is given by the representable functor from the category of commutative k-algebras to the category of groups:

$$G(\,\text{–}\,) \,:\, \mathtt{CommAlg}_k \,\to\, \mathtt{Gr}\,, \quad A \mapsto G(A)\,. \tag{4.2.1}$$

The algebra $\mathcal{O}(G)$ that represents (4.2.1) is called the coordinate ring of G: it is a commutative Hopf algebra with coproduct $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$, $f \mapsto f_{(1)} \otimes f_{(2)}$, corresponding to the multiplication in G. The functor (4.2.1) has a left adjoint functor which we denote by

$$(\,\text{–}\,)_G \,:\, \mathtt{Gr} \to \mathtt{CommAlg}_k\,, \quad \Gamma \mapsto \Gamma_G\,. \tag{4.2.2}$$

We call (4.2.2) the *representation functor* in G, and think of the commutative algebra Γ_G assigned to a discrete group Γ as the coordinate ring $\mathcal{O}[\mathrm{Rep}_G(\Gamma)]$ of the affine scheme $\mathrm{Rep}_G(\Gamma)$ parametrizing the representations of Γ in G.

It is easy to see that the algebra Γ_G has the following explicit presentation

$$\Gamma_G = \mathrm{Sym}_k[\,k[\Gamma] \otimes_k \mathcal{O}(G)\,]/\langle\!\langle R \rangle\!\rangle\,,$$

where the relations R are given by

$$\gamma \otimes f_1 f_2 - (\gamma \otimes f_1) \cdot (\gamma \otimes f_2)\,, \quad \forall\, \gamma \in \Gamma\,,\ \forall\, f_1\,, f_2 \in \mathcal{O}(G)\,,$$
$$\gamma_1\gamma_2 \otimes f - (\gamma_1 \otimes f_{(1)}) \cdot (\gamma_2 \otimes f_{(2)})\,, \quad \forall\, \gamma_1\,, \gamma_2 \in \Gamma\,,\ \forall\, f \in \mathcal{O}(G)\,,$$
$$\gamma \otimes e_G - 1\,, \quad e_\Gamma \otimes f - f(e_G) \cdot 1\,, \quad \forall\, \gamma \in \Gamma\,,\ \forall\, f \in \mathcal{O}(G)\,.$$

Now, to construct the derived representation functor we embed the category of groups into the category $\mathtt{sGr}$ of simplicial groups and extend the functor (4.2.2) to $\mathtt{sGr}$ in the natural way, assigning to a simplicial group $\Gamma_* : \Delta^{\rm op} \to \mathtt{Gr}$ the simplicial commutative algebra $(\Gamma_*)_G : \Delta^{\rm op} \to \mathtt{Gr} \to \mathtt{CommAlg}_k$. We will keep the notation $(\,\text{–}\,)_G$ for this extended representation functor:

$$(\,\text{–}\,)_G :\ \mathtt{sGr} \to \mathtt{sCommAlg}_k\,. \tag{4.2.3}$$

Lemma 4.2.4. *The functor* (4.2.3) *maps weak equivalences between cofibrant objects in* $\mathtt{sGr}$ *to weak equivalences in* $\mathtt{sCommAlg}_k$*, and hence has a total left derived functor*

$$\boldsymbol{L}(\,\text{–}\,)_G :\ Ho(\mathtt{sGr}) \ \to\ Ho(\mathtt{sCommAlg}_k)\,. \tag{4.2.5}$$

Proof. Suppose that $f : \Gamma \to \Gamma'$ is a weak equivalence between cofibrant simplicial groups. Since sGr is a fibrant model category, Γ and Γ' are both fibrant-cofibrant objects. By Whitehead's Theorem, the map f has then a homotopy inverse $g : \Gamma' \to \Gamma$, such that $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$. Now, any homotopy between fibrant-cofibrant objects can be realized using a good cylinder object in $\mathtt{sGr}$. Since sGr is a simplicial model category, there is a natural choice of good cylinder objects for Γ and Γ': namely, $\Gamma \sqcup \Gamma \to \Gamma \times \Delta[1] \to \Gamma$, and similarly for Γ'. For such cylinder objects, the simplicial homotopies (see [112, Def. 5.1]) can be defined by explicit combinatorial relations which are obviously preserved by the functor $(\,\text{–}\,)_G$. Thus, we conclude that the morphism $g_G : \Gamma'_G \to \Gamma_G$ is a homotopy inverse of $f_G : \Gamma_G \to \Gamma'_G$ in $\mathtt{sCommAlg}_k$ and hence f_G and g_G are mutually inverse isomorphisms in $\mathrm{Ho}(\mathtt{sCommAlg}_k)$. The existence of the derived functor (4.2.5) follows now from [47, Prop. 9.3]. $\square$

Remark 4.2.6. Despite the fact that the representation functor (4.2.3) is a left adjoint functor, it is *not* left Quillen. Indeed, by definition, any left Quillen functor preserves cofibrations; in particular, maps cofibrant objects to cofibrant. To see that this is not the case for (4.2.3) take $G = \mathbb{G}_m$, the multiplicative group, and apply (4.2.3) to the free group on one generator $\Gamma = \mathbb{F}_1$, viewed as a discrete simplicial group in $\mathtt{sGr}$. The result is $\Gamma_G \cong k[x, x^{-1}]$, which is *not* a cofibrant simplicial algebra in $\mathtt{sCommAlg}_k$. Thus, the simplicial adjunction $(-)_G : \mathtt{sGr} \rightleftarrows \mathtt{sCommAlg}_k : G(-)$ is *not* a Quillen pair, so the result of Lemma 4.2.4 is not automatic.

For a fixed simplicial group $\Gamma \in \mathtt{sGr}$, we define $\mathrm{DRep}_G(\Gamma) := \boldsymbol{L}(\Gamma)_G$ and call $\mathrm{DRep}_G(\Gamma)$ the *derived representation scheme* of Γ in the algebraic group G. By definition, $\mathrm{DRep}_G(\Gamma)$ is a simplicial commutative algebra viewed as an object of $\mathrm{Ho}(\mathtt{sCommAlg}_k)$. Following [23, 21], we refer to the corresponding homotopy groups as the *representation homology* of Γ in G:

$$\pi_*[\mathrm{DRep}_G(\Gamma)] := \mathrm{H}_*(N[\mathrm{DRep}_G(\Gamma)])$$

It is easy to check that the functor (4.2.3) commutes with π_0, so that for any $\Gamma \in \mathtt{sGr}$, there is a natural isomorphism in $\mathtt{CommAlg}_k$:

$$\pi_0[\mathrm{DRep}_G(\Gamma)] \cong [\pi_0(\Gamma)]_G \tag{4.2.7}$$

Hence, for an ordinary group $\Gamma \in \mathtt{Gr}$ (viewed as a discrete simplicial group), we have

$$\pi_0[\mathrm{DRep}_G(\Gamma)] \cong \Gamma_G = \mathcal{O}[\mathrm{Rep}_G(\Gamma)] \,.$$

This justifies our notation and terminology for $\mathrm{DRep}_G(\Gamma)$. We record one useful property of the derived representation functor that will play a role in computations in Section 7.

Lemma 4.2.8. *The functor* $\mathrm{DRep}_G : \mathit{Ho}(\mathtt{sGr}) \to \mathit{Ho}(\mathtt{sCommAlg}_k)$ *preserves homotopy pushouts.*

The proof of Lemma 4.2.8 requires some technical results about simplicial commutative algebras; we therefore defer it to Section 11 (see Proposition 11.3.14).

Representation homology of spaces

We now define the representation homology of a (reduced) simplicial set by mimicking Pirashvili's definition of higher Hochshild homology. Our starting point is the known fact (see, e.g., [134, Sect. 5]) that the category of commutative Hopf algebras over a field k is equivalent to the category of k-algebras of the PROP of finitely generated free groups. To be precise, let $\mathfrak{G}$ denote the full subcategory of $\mathtt{Gr}$ whose objects the free groups based on the sets $\underline{n} = \{1, 2, \ldots, n\}$ for $n \geq 0$. We denote such groups by $\langle n \rangle := \mathbb{F}\langle \underline{n} \rangle$ (where, by convention, $\langle 0 \rangle$ is the identity group) and write $k\langle n \rangle$ for the corresponding group algebras over k. The category $\mathfrak{G}$ is a PROP, with monoidal product $\boxtimes$ being the free product of groups, so that $\langle n \rangle \boxtimes \langle m \rangle = \langle n + m \rangle$. A commutative Hopf algebra $\mathcal{H}$ over k defines the (strong monoidal) covariant functor $\mathfrak{G} \to \mathtt{Vect}_k$, $\langle n \rangle \mapsto \mathcal{H}^{\otimes n}$, which we denote by $\underline{\mathcal{H}}$. The assignment $\mathcal{H} \mapsto \underline{\mathcal{H}}$ gives an equivalence between the category of commutative Hopf algebras over k and the category of k-algebras over the PROP $\mathfrak{G}$. Dually, the category of *cocommutative* Hopf algebras is equivalent to the category k-algebras over the opposite PROP $\mathfrak{G}^{\mathrm{op}}$.

Now, observe that for any commutative Hopf algebra $\mathcal{H}$, the functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathtt{Vect}_k$ takes values in the category of commutative algebras, *i.e.*, it can be viewed as a functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathtt{CommAlg}_k$. We extend this last functor to the category $\mathtt{FGr}$ of all free groups by taking the left Kan extension along the inclusion $\mathfrak{G} \hookrightarrow \mathtt{FGr}$. To be precise, let $\mathtt{FGr}$ denote the category of *based* free groups whose objects are pairs (Γ, S), where $\Gamma = \langle S \rangle$ is a free group with a specified generating set S, and morphisms are arbitrary group homomorphisms $\Gamma \to \Gamma'$ (not necessarily, preserving the generating sets). We have the natural inclusion functor $i : \mathfrak{G} \hookrightarrow \mathtt{FGr}$ that takes $\langle n \rangle$ to $(\langle n \rangle, \underline{n})$. The Kan extension of $\underline{\mathcal{H}}$ along i then defines a functor $\mathtt{FGr} \to \mathtt{CommAlg}_k$ that assigns to the free group $\langle S \rangle$ on a set S the commutative algebra $S \otimes \mathcal{H} = \otimes_{s \in S} \mathcal{H}_s$ (*cf.* (3.2.1)). We continue to denote this functor by $\underline{\mathcal{H}}$.

Let X be a reduced simplicial set (or equivalently, a connected pointed CGWH topological space). Recall that the Kan loop group construction gives a functor $\mathbb{G}X : \Delta^{\mathrm{op}} \to \mathtt{FGr}$ that takes $[n] \in \Delta^{\mathrm{op}}$ to the free group $\mathbb{G}X_n = \langle B_n \rangle$ based on the set $B_n = X_{n+1} \setminus s_0(X_n)$. Now, given a commutative Hopf algebra $\mathcal{H}$, we consider the composition of functors

$$\Delta^{\mathrm{op}} \xrightarrow{\mathbb{G}X} \mathtt{FGr} \xrightarrow{\underline{\mathcal{H}}} \mathtt{CommAlg}_k \,,$$

which defines a simplicial commutative algebra $\underline{\mathcal{H}}(\mathbb{G}X)$.

Definition 4.2.9. The *representation homology* of X in $\mathcal{H}$ is defined by

$$\mathrm{HR}_*(X, \mathcal{H}) \,:=\, \pi_*[\underline{\mathcal{H}}(\mathbb{G}X)] \,=\, \mathrm{H}_*[N(\underline{\mathcal{H}}(\mathbb{G}X))]\,. \tag{4.2.10}$$

If G is an affine group scheme with coordinate ring $\mathcal{H} = \mathcal{O}(G)$, we also write $\mathrm{HR}_*(X, G)$ for $\mathrm{HR}_*(X, \mathcal{H})$. Clearly, a morphism $f : X \to Y$ of reduced simplicial sets induces a map of graded commutative algebras $\mathrm{HR}_*(f,\mathcal{H})$: $\mathrm{HR}_*(X,\mathcal{H}) \to \mathrm{HR}_*(Y,\mathcal{H})$. Thus, representation homology defines a covariant functor $\mathrm{HR}(-,\mathcal{H}) : \mathtt{sSet}_0 \to \mathtt{GrComm\,Alg}_k$.

The following proposition justifies the above definition of representation homology.

Proposition 4.2.11. *For any $X \in \mathtt{sSet}_0$, there is a natural isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(X,G) \cong \pi_*[\mathrm{DRep}_G(\mathbb{G}X)]\,. \tag{4.2.12}$$

In particular, $\mathrm{HR}_0(X,G) \cong \pi_1(X)_G$*, where $\pi_1(X)$ is the fundamental group of X.*

Proof. If $\mathcal{H} = \mathcal{O}(G)$, we have natural isomorphisms $\underline{\mathcal{H}}(\langle S\rangle) \cong \bigotimes_{s\in S}\mathcal{O}(G)_s \cong (\langle S\rangle)_G$ for any set S. This implies that $\underline{\mathcal{H}}(\mathbb{G}X) \cong (\mathbb{G}X)_G$ in $s\mathtt{CommAlg}_k$. On the other hand, by Proposition 4.1.3, the simplicial group $\mathbb{G}X$ is semi-free, and hence a cofibrant object in $\mathtt{sGr}$. This implies that $\boldsymbol{L}(\mathbb{G}X)_G \cong (\mathbb{G}X)_G$ in $\mathrm{Ho}(s\mathtt{CommAlg}_k)$. Thus, $\mathrm{DRep}_G(\mathbb{G}X)$ is represented by the simplical commutative algebra $\underline{\mathcal{H}}(\mathbb{G}X)$, so $\pi_*[\mathrm{DRep}_G(\mathbb{G}X)] \cong \pi_*[\underline{\mathcal{H}}(\mathbb{G}X)]$. The isomorphism for $\mathrm{HR}_0(X,G)$ is obtained by composing (4.2.12) with (4.2.7) and the natural isomorphism of groups $\pi_0(\mathbb{G}X) \cong \pi_1(X)$. $\square$

Let Γ be a discrete group, and let $X = \mathrm{B}\Gamma$ be the classifying space (*i.e.*, the simplicial nerve) of Γ. As a simple application of Proposition 4.2.11, we get

Corollary 4.2.13. $\mathrm{HR}_*(\mathrm{B}\Gamma,G) \cong \pi_*[\mathrm{DRep}_G(\Gamma)]$. *In particular,* $\mathrm{HR}_0(\mathrm{B}\Gamma,G) \cong \Gamma_G$.

Proof. The Kan adjunction (4.1.2) gives the canonical cofibrant resolution $\mathbb{G}\overline{W}\Gamma \xrightarrow{\sim} \Gamma$ in $\mathtt{sGr}$. Since Γ is discrete, we have $\overline{W}\Gamma = \mathrm{B}\Gamma$, and the result follows from Proposition 4.2.11. $\square$

Corollary 4.2.14. *For any $X, Y \in \mathtt{sSet}_0$, there is a natural isomorphism*

$$\mathrm{HR}_*(X \vee Y, G) \cong \mathrm{HR}_*(X, G) \otimes \mathrm{HR}_*(Y, G) .$$

Proof. Recall that the wedge sum is a (categorical) coproduct in $\mathtt{sSet}_0$. Since $\mathbb{G}$ is a left adjoint functor, we have $\mathbb{G}(X \vee Y) \cong \mathbb{G}X * \mathbb{G}Y$. By Lemma 4.2.8, it follows that $\mathrm{DRep}_G(\mathbb{G}X * \mathbb{G}Y) \cong \mathrm{DRep}_G(\mathbb{G}X) \otimes \mathrm{DRep}_G(\mathbb{G}Y)$. The desired result is now immediate from Künneth's Theorem and Proposition 4.2.11. $\square$

The fundamental spectral sequence

Now, we introduce the functor categories $\mathfrak{G}$-$\mathtt{Mod}$ and $\mathtt{Mod}$-$\mathfrak{G}$, whose objects are all covariant (resp., contravariant) functors from $\mathfrak{G}$ to the category of vector spaces. We regard these objects as left and right modules over $\mathfrak{G}$, respectively. Both categories are abelian with sufficiently many projective and injective objects. There is a natural bifunctor

$$- \otimes_{\mathfrak{G}} - \,:\, \mathtt{Mod}\text{-}\mathfrak{G} \times \mathfrak{G}\text{-}\mathtt{Mod} \to \mathtt{Vect}_k$$

which is right exact with respect to each argument, preserves sums and is left balanced in the sense of [42]. Explicitly, this bifunctor can be defined by formula (3.3.1) with $\mathfrak{F}$ replaced by $\mathfrak{G}$.

Since $- \otimes_{\mathfrak{G}} -$ is left balanced, the derived functors with respect to each argument are naturally isomorphic, and we denote their common value by $\mathrm{Tor}_*^{\mathfrak{G}}(-,-)$. Note that for any left $\mathfrak{G}$-module $\mathcal{M}$, the functor $- \otimes_{\mathfrak{G}} \mathcal{M} : \mathtt{Mod}$-$\mathfrak{G} \to \mathtt{Vect}_k$ is left adjoint to the functor $\underline{\mathrm{Hom}}(\mathcal{M}, -) : \mathtt{Vect}_k \to \mathtt{Mod}$-$\mathfrak{G}$, where $\underline{\mathrm{Hom}}(\mathcal{M}, V)$ is the right $\mathfrak{G}$-module $\langle n \rangle \mapsto \mathrm{Hom}_k(\mathcal{M}(\langle n \rangle), V)$ for any vector space V. Similarly, for any right $\mathfrak{G}$-module $\mathcal{N}$, the functor $\mathcal{N} \otimes_{\mathfrak{G}} -$ is left adjoint to the functor $\underline{\mathrm{Hom}}(\mathcal{N}, -) : \mathtt{Vect}_k \to \mathfrak{G}\text{-}\mathtt{Mod}$. Hence, both functors $-\otimes_{\mathfrak{G}} \mathcal{M}$ and $\mathcal{N} \otimes_{\mathfrak{G}} -$ commute with colimits.

To state our first theorem we need some notation. First, we recall that if Γ is any group, $k[\Gamma]$ is a cocommutative Hopf algebra: thus, $k[\Gamma]$ defines a right $\mathfrak{G}$-module in $\mathtt{Mod}\text{-}\mathfrak{G}$. Now, if X is a reduced simplicial set, $k[\mathbb{G}X]$ defines a simplicial right $\mathfrak{G}$-module in $\mathtt{sMod}\text{-}\mathfrak{G}$. Applying the normalization functor $\underline{N} : \mathtt{sMod}\text{-}\mathfrak{G} \to \mathtt{Ch}_{\geq 0}(\mathtt{Mod}\text{-}\mathfrak{G})$ to this simplicial module, we get a chain complex of $\mathfrak{G}$-modules and hence an object in the derived category $\mathscr{D}(\mathtt{Mod}\text{-}\mathfrak{G})$. Abusing notation, we will denote this object by $\underline{N}(k[\mathbb{G}X])$.

Theorem 4.2.15. *For any $X \in \mathtt{sSet}_0$ and any commutative Hopf algebra $\mathcal{H}$, there is a natural isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(X, \mathcal{H}) \cong \mathrm{H}_*[\underline{N}(k[\mathbb{G}X]) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \underline{\mathcal{H}}] \,.$$

To prove Theorem 4.2.15 we need a simple lemma. Recall that for $n \geq 0$, we denote by $k\langle n \rangle$ the group algebra of the free group based on the set $\underline{n} = \{1, 2, \ldots, n\}$. Regarding it as a cocommutative Hopf algebra, we get a right $\mathfrak{G}$-module which (to simplify the notation) we also denote by $k\langle n \rangle$.

Lemma 4.2.16. *For each $n \geq 0$, the $\mathfrak{G}$-module $k\langle n \rangle$ is a projective object in $\mathtt{Mod}\text{-}\mathfrak{G}$.*

Proof. For a fixed $n \geq 0$, let $h^n := k[\mathrm{Hom}_{\mathfrak{G}}(-, \langle n \rangle)]$ denote the standard right $\mathfrak{G}$-module associated to the object $\langle n \rangle \in \mathfrak{G}$. This is a projective object in $\mathtt{Mod}\text{-}\mathfrak{G}$. Indeed, by Yoneda Lemma, there is a natural isomorphism $\mathrm{Hom}_{\mathtt{Mod}\text{-}\mathfrak{G}}(h^n, \mathcal{N}) \cong \mathcal{N}(\langle n \rangle)$ for any $\mathcal{N} \in \mathtt{Mod}\text{-}\mathfrak{G}$. The sequence of $\mathfrak{G}$-modules $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ is exact in $\mathtt{Mod}\text{-}\mathfrak{G}$ if and only if the sequence of k-vector spaces $0 \to \mathcal{N}'(\langle n \rangle) \to \mathcal{N}(\langle n \rangle) \to \mathcal{N}''(\langle n \rangle) \to 0$ is exact for all $n \geq 0$. It follows that $\mathrm{Hom}_{\mathtt{Mod}\text{-}\mathfrak{G}}(h^n, -)$:

$\mathtt{Mod}\text{-}\mathfrak{F} \to \mathtt{Vect}_k$ is an exact functor, and hence h^n is a projective object in $\mathtt{Mod}\text{-}\mathfrak{G}$. On the other hand, for any $m \geq 0$, we have

$$h^n(\langle m \rangle) \;=\; k[\mathrm{Hom}_{\mathfrak{G}}(\langle m \rangle, \langle n \rangle)] \;\cong\; k[\langle n \rangle^{\times m}] \;\cong\; [k\langle n \rangle]^{\otimes m} \;=\; k\langle n \rangle(\langle m \rangle)\;,$$

which shows that $k\langle n \rangle \cong h^n$ as right $\mathfrak{G}$-modules. This finishes the proof of the lemma. $\square$

Proof of Theorem 4.2.15. By Lemma 4.2.16, for any $n \geq 0$, $k\langle n \rangle$ is a projective right $\mathfrak{G}$-module such that $k\langle n \rangle \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \;\cong\; \underline{\mathcal{H}}(\langle n \rangle)$. Since colimits of projective modules are flat and commute with left Kan extensions, this implies that $k\langle S \rangle$ is a *flat* right $\mathfrak{G}$-module and $k\langle S \rangle \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \;\cong\; \underline{\mathcal{H}}(\langle S \rangle)$ for any set S. Extending the last isomorphism levelwise to simplicial sets, we get an isomorphism of simplicial vector spaces $k[\mathbb{G}X] \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \;\cong\; \underline{\mathcal{H}}(\mathbb{G}X)$. Further, since each $k[\mathbb{G}X_n]$ is a flat right $\mathfrak{G}$-module, the normalized chain complex $\underline{N}(k[\mathbb{G}X])$ is a complex of flat $\mathfrak{G}$-modules, hence we have a natural isomorphism in the derived category $\mathscr{D}(\mathtt{Mod}\text{-}\mathfrak{G})$:

$$\underline{N}(k[\mathbb{G}X]) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \underline{\mathcal{H}} \;\cong\; \underline{N}(k[\mathbb{G}X]) \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \;\cong\; N(\underline{\mathcal{H}}(\mathbb{G}X))\,.$$

At the homology level, this induces the desired isomorphism of Theorem 4.2.15. $\square$

For our next theorem we recall that the singular chain complex $C_*(\Omega X; k)$ of the (Moore) loop space ΩX of a pointed topological space X has a natural structure of a cocommutative DG Hopf algebra. The coproduct on $C_*(\Omega X; k)$ is induced by the Alexander-Whitney diagonal, while the product comes from the structure of a topological monoid on ΩX via the Eilenberg-Zilber map (see, e.g., [58, Section 26]). Thus, the homology $\mathrm{H}_*(\Omega X; k)$ of ΩX is a graded cocommutative Hopf algebra called the *Pontryagin algebra* of X.

Now, any graded cocommutative Hopf algebra H defines a graded right $\mathfrak{G}$-module $\underline{\mathrm{H}}$ (*i.e.*, a contravariant functor from $\mathfrak{G}$ to the category of graded vector spaces). For $q \in \mathbb{Z}$, we let $\underline{\mathrm{H}}_q$ denote the graded component of $\underline{\mathrm{H}}$ of degree q; thus, $\underline{\mathrm{H}}_q : \mathfrak{G}^{\mathrm{op}} \to \mathtt{Vect}_k$ is a right $\mathfrak{G}$-module that assigns $\langle n \rangle \mapsto [\mathrm{H}^{\otimes n}]_q$, the q-th graded component of the graded vector space $\mathrm{H}^{\otimes n}$. Note that the $\mathfrak{G}$-module $\underline{\mathrm{H}}_q$ depends on *all* graded components of the Hopf algebra H, and not solely on H_q. With this notation, we can now state our second theorem, which is an analogue of [133, Theorem 2.4] for representation homology.

Theorem 4.2.17. *There is a natural first quadrant spectral sequence*

$$E^2_{pq} \;=\; \mathrm{Tor}^{\mathfrak{G}}_p(\underline{\mathrm{H}}_q(\Omega X; k), \underline{\mathcal{H}}) \;\underset{p}{\Longrightarrow}\; \mathrm{HR}_n(X, \mathcal{H}) \tag{4.2.18}$$

converging to the representation homology of X.

Proof. Recall from the proof of Theorem 4.2.15 that $\underline{N}(k[\langle X \rangle])$ is a non-negatively graded chain complex of flat right $\mathfrak{G}$-modules. Hence, for any left $\mathfrak{G}$-module $\underline{\mathcal{H}}$, there is a standard 'Hypertor' spectral sequence (see, e.g., [164, Application 5.7.8]):

$$E^2_{pq} \;=\; \mathrm{Tor}^{\mathfrak{G}}_p(\mathrm{H}_q[\underline{N}(k[\mathbb{G}X])],\, \underline{\mathcal{H}}) \;\underset{p}{\Longrightarrow}\; \mathrm{H}_{p+q}\,[\underline{N}(k[\mathbb{G}X]) \otimes_{\mathfrak{G}} \underline{\mathcal{H}}]\;.$$

By Theorem 4.2.15, the limit of this spectral sequence is isomorphic to $\mathrm{HR}_*(X, \mathcal{H})$. To prove the theorem we need only to show that $\mathrm{H}_*[\underline{N}(k[\mathbb{G}X])] \cong \underline{\mathrm{H}}_*(\Omega X; k)$ as graded right $\mathfrak{G}$-modules.

By Kan's Theorem 4.1.4, $|\mathbb{G}X|$ is a topological group, which is weakly equivalent, as an H-space, to the based loop space ΩX. This implies, in particular, that $\mathrm{H}_*[N(k[\mathbb{G}X])] \;\cong\; \mathrm{H}_*(\Omega X; k)$ as graded Hopf algebras, and hence $\underline{\mathrm{H}}_*[N(k[\mathbb{G}X])] \;\cong\; \underline{\mathrm{H}}_*(\Omega X; k)$ as graded $\mathfrak{G}$-modules. Note that $N(k[\mathbb{G}X])$

stands here for the normalized chain complex of the simplicial Hopf algebra $k[\mathbb{G}X]$, while $\underline{N}(k[\mathbb{G}X])$ in the above spectral sequence denotes the normalized chain complex of the simplicial $\mathfrak{G}$-module $k[\mathbb{G}X]$. We need to check that $\mathrm{H}_*[\underline{N}(k[\mathbb{G}X])] \cong \underline{\mathrm{H}}_*[N(k[\mathbb{G}X])]$ as graded $\mathfrak{G}$-modules. Now, the simplicial $\mathfrak{G}$-module $k[\mathbb{G}X]$ assigns to $\langle m\rangle \in \mathfrak{G}$ the simplicial vector space $k[\mathbb{G}X_*]^{\otimes m} = \{k[\mathbb{G}X_n]^{\otimes m}\}_{n\geq 0}$. By the Eilenberg-Zilber Theorem, the normalized chain complex of this simplicial vector space is homotopy equivalent to $N(k[\mathbb{G}X])^{\otimes m}$, while, by Kunneth's formula, the homology of $N(k[\mathbb{G}X])^{\otimes m}$ is naturally isomorphic to $\mathrm{H}_*[N(k[\mathbb{G}X])]^{\otimes m}$. This shows that $\mathrm{H}_*(\underline{N}(k[\mathbb{G}X]))(\langle m\rangle) \cong \mathrm{H}_*[N(k[\mathbb{G}X])]^{\otimes m}$ for any $m \geq 0$, completing the proof of the theorem. □

Theorem 4.2.17 has several interesting implications. First, we consider one important special case when the spectral sequence (4.2.18) collapses at E^2-term.

Corollary 4.2.19. *Let Γ be a discrete group. Then, for any affine algebraic group G, there is a natural isomorphism*

$$\mathrm{HR}_*(\mathrm{B}\Gamma, G) \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \mathcal{O}(G)) \,.$$

In particular, $\mathrm{HR}_0(\mathrm{B}\Gamma, G) \cong k[\Gamma] \otimes_{\mathfrak{G}} \mathcal{O}(G)$.

Proof. The classifying space $X = \mathrm{B}\Gamma$ is an Eilenberg-MacLane space of type $K(\Gamma, 1)$. Its loop space ΩX is homotopy equivalent to Γ, where Γ is considered as a discrete topological space. Hence, $\mathrm{H}_q(\Omega X; k) = 0$ for all $q > 0$, while $\mathrm{H}_0(\Omega X; k) \cong k[\Gamma]$ as a Hopf algebra. Thus, for $X = \mathrm{B}\Gamma$, the spectral sequence (4.2.18) collapses on the p-axis, giving the required isomorphism. □

Combining the isomorphisms of Corollary 4.2.13 and Corollary 4.2.19, we can express the representation homology of Γ (originally defined as a non-abelian derived functor) in terms of classical abelian homological algebra:

$$\pi_*[\mathrm{DRep}_G(\Gamma)] \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \mathcal{O}(G)) \ .$$

In degree 0, we have a natural isomorphism expressing the coordinate ring of the representation variety $\mathrm{Rep}_G(\Gamma)$ as a functor tensor product:

$$\mathcal{O}[\mathrm{Rep}_G(\Gamma)] \cong k[\Gamma] \otimes_{\mathfrak{G}} \mathcal{O}(G) \ .$$

This last isomorphism was found in [100], and it was one of the starting points for Part I of this thesis.

The result of Theorem 4.2.17 holds for any (not necessarily, monoidal) left $\mathfrak{G}$-module. In particular, if we take a *reductive* affine algebraic group G and define a left $\mathfrak{G}$-module $\mathcal{O}(G)^G \in \mathfrak{G}\text{-}\mathtt{Mod}$ by the formula $\langle n \rangle \mapsto [\mathcal{O}(G)^{\otimes n}]^G = \mathcal{O}(G\times \overset{n}{\dots} \times G)^G$, then, for any $X \in \mathtt{sSet}_0$, we obtain a homology spectral sequence

$$E^2_{pq} \ = \ \mathrm{Tor}_p^{\mathfrak{G}}(\underline{\mathrm{H}}_q(\Omega X; k),\ \mathcal{O}(G)^G) \Longrightarrow \mathrm{HR}_n(X,\ G)^G \qquad (4.2.20)$$

converging to the G-invariant part of representation homology of X. The proof of Corollary 4.2.19 shows that, for $X = \mathrm{B}\Gamma$, the spectral sequence (4.2.20) collapses on the p-axis, giving an isomorphism

$$\mathrm{HR}_*(\mathrm{B}\Gamma, G)^G \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \mathcal{O}(G)^G) \ .$$

In degree 0, we therefore have $\mathcal{O}[\mathrm{Rep}_G(\Gamma)]^G \cong k[\Gamma] \otimes_{\mathfrak{G}} \mathcal{O}(G)^G$.

To state futher consequences of Theorem 4.2.17 we introduce some terminology. We will say that a map $f : X \to Y$ of pointed topological spaces is a *Pontryagin equivalence* (over k) if it induces an isomorphism $\mathrm{H}_*(\Omega X; k) \cong \mathrm{H}_*(\Omega Y; k)$ of Pontryagin algebras (or equivalently, a quasi-isomorphism $C_*(\Omega X; k) \overset{\sim}{\to} C_*(\Omega Y; k)$ of DG Hopf algebras). The next result is obtained by applying to

(4.2.18) a standard comparison theorem for homology spectral sequences (see [164, Theorem 5.1.12]).

Corollary 4.2.21. *If $f : X \to Y$ is a Pontryagin equivalence, the induced map on representation homology $f_* : \mathrm{HR}_*(X, \mathcal{H}) \xrightarrow{\sim} \mathrm{HR}_*(Y, \mathcal{H})$ is an isomorphism for any Hopf algebra $\mathcal{H}$.*

We remark that Corollary 4.2.21 does not say that an *arbitrary* isomorphism of Hopf algebras $\mathrm{H}_*(\Omega X; k) \cong \mathrm{H}_*(\Omega Y; k)$ gives an isomorphism $\mathrm{HR}_*(X, \mathcal{H}) \cong \mathrm{HR}_*(Y, \mathcal{H})$. (Indeed, an abstract isomorphism of Pontryagin algebras need not even induce a map on representation homology.) Still, Corollary 4.2.19 shows that if both X and Y are aspherical spaces, then any isomorphism of Pontryagin algebras induces an isomorphism on representation homology.

Next, we recall that the singular chain complex $C_*(X; k)$ of any space X is naturally a DG coalgebra with comultiplication defined by the Alexander-Whitney diagonal. Moreover, if X is path-connected, there is a quasi-isomorphism of DG coalgebras (see [59, Theorem 6.3])

$$C_*(X; k) \simeq \mathbb{B}[C_*(\Omega X; k)] ,$$

where $\mathbb{B}$ is the classical bar construction. Since $\mathbb{B}$ preserves quasi-isomorphisms, any Pontryagin equivalence $f : X \to Y$ of path-connected spaces is necessarily a homology equivalence, *i.e.*, it induces an isomorphism on singular homology $\mathrm{H}_*(X; k) \xrightarrow{\sim} \mathrm{H}_*(Y; k)$. The converse is not always true unless X and Y are simply-connected. In the latter case, we have the following well-known result (*cf.* [136, Part I, Prop. 1.1]).

Lemma 4.2.22. *Let $f : X \to Y$ be a map of simply-connected pointed topological spaces. The following conditions are equivalent:*

(1) f is a rational homology equivalence: i.e. $f_* : \mathrm{H}_*(X;\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*(Y;\mathbb{Q})$;

(2) f is a rational Pontryagin equivalence: i.e. $f_* : \mathrm{H}_*(\Omega X;\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*(\Omega Y;\mathbb{Q})$;

(3) f is a rational homotopy equivalence: i.e. $f_* : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. The equivalence $(1) \Leftrightarrow (2)$ follows a classical theorem of Adams [2] that asserts that, for any simply-connected space X, there is a quasi-isomorphism of DG algebras: $C_*(\Omega X;k) \simeq \mathbf{\Omega}[C_*(X;k)]$, where $\mathbf{\Omega}$ is the cobar construction.

To prove that $(2) \Leftrightarrow (3)$ we first recall that, for any simply-connected X, the $\mathbb{Q}$-vector space $L_X := \pi_*(\Omega X)_{\mathbb{Q}} \cong \pi_{*+1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ carries a natural bracket (called the Whitehead product) making it a graded Lie algebra[1]. Thus, a map $f : X \to Y$ is a rational homotopy equivalence if and only if it induces an isomorphism of Lie algebras $f_* : L_X \to L_Y$. Then, a classical theorem of Milnor and Moore (see [58, Theorem 21.5]) implies that the Hurewicz homomorphism $\pi_*(\Omega X) \to \mathrm{H}_*(\Omega X;\mathbb{Q})$ induces an isomorphism of graded Hopf algebras $UL_X \xrightarrow{\sim} \mathrm{H}_*(\Omega X;\mathbb{Q})$, where $U(L_X)$ is the universal enveloping algebra of L_X. This yields the equivalence $(2) \Leftrightarrow (3)$. □

We say that a map $f : X \to Y$ of simply-connected spaces is a *rational homotopy equivalence* if the equivalent conditions of Lemma 4.2.22 hold.

Proposition 4.2.23. *A rational homotopy equivalence induces an isomorphism on representation homology. Thus, $\mathrm{HR}_*(X, \mathcal{H})$ depends only on the rational homotopy type of X.*

Proof. By Lemma 4.2.22(2), a rational homotopy equivalence $X \to Y$ induces an isomorphism $\mathrm{H}_*(\Omega X;\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*(\Omega Y;\mathbb{Q})$. Since $\mathrm{char}(k) = 0$, we have $\mathbb{Q} \subseteq k$,

[1] The Lie algebra L_X is called the *homotopy Lie algebra* of X.

and the Universal Coefficient Theorem implies that $\mathrm{H}_*(\Omega X;k) \cong \mathrm{H}_*(\Omega Y;k)$. The claim then follows from Corollary 4.2.21. $\square$

Next, we look at higher connected spaces. Recall that a space X is said to be *n-connected* if X is path-connected and its first n homotopy groups vanish, *i.e.* $\pi_i(X) = 0$ for $1 \leq i \leq n$.

Proposition 4.2.24. *Let X be an n-connected space for some $n \geq 1$, and let $\mathcal{H} = \mathcal{O}(G)$. Then*

$$\mathrm{HR}_q(X, \mathcal{H}) = \begin{cases} k & \text{for} \quad q = 0 \\ 0 & \text{for} \quad 1 \leq q < n \\ \mathrm{H}_{q+1}(X; \mathfrak{g}^*) & \text{for} \quad n \leq q < 2n \end{cases} \tag{4.2.25}$$

where $\mathfrak{g} := \mathrm{Lie}(G)$ is the Lie algebra of G and $\mathfrak{g}^*$ is its k-linear dual.

Proof. If a space X is n-connected, its homotopy Lie algebra $L_X = \pi_*(\Omega X)_{\mathbb{Q}} \cong \pi_{*+1}(X)_{\mathbb{Q}}$ is n-reduced, *i.e.* $(L_X)_q = 0$ for $0 \leq q \leq n-1$. Since $\mathrm{H}_*(\Omega X;\mathbb{Q}) \cong UL_X$ and $\mathbb{Q} \subseteq k$, we have $\mathrm{H}_0(\Omega X;k) \cong k$, $\mathrm{H}_q(\Omega X;k) = 0$ for $1 \leq q \leq n-1$, and

$$\mathrm{H}_q(\Omega X;k) \cong (L_X)_q \otimes_{\mathbb{Q}} k \cong \pi_{q+1}(X)_k \cong \mathrm{H}_{q+1}(X;k) \quad \text{for} \quad n \leq q \leq 2n-1\,,$$

where the last isomorphism is a consequence of the Rational Hurewicz Theorem (see, e.g., [99]).

Now, recall that for a fixed $q \geq 0$, the right $\mathfrak{G}$-module $\underline{\mathrm{H}}_q(\Omega X;k)$ is defined as the functor $\mathfrak{G}^{\mathrm{op}} \to \mathtt{Vect}_k$, $\langle m \rangle \mapsto [\mathrm{H}_*(\Omega X;k)^{\otimes m}]_q$. It follows from this definition that

$$\underline{\mathrm{H}}_q(\Omega X;k) = \begin{cases} \underline{k} & \text{for} \quad q = 0 \\ 0 & \text{for} \quad 1 \leq q \leq n-1 \\ \mathtt{lin}_k^* \otimes \mathrm{H}_{q+1}(X;k) & \text{for} \quad n \leq q \leq 2n-1 \end{cases}$$

where $\mathtt{lin}_k$ is the linearization functor:

$$\mathtt{lin}_k : \mathfrak{G} \to \mathtt{Vect}_k\ , \quad \langle m\rangle \mapsto \langle m\rangle_{\rm ab}\otimes_{\mathbb{Z}} k = k^{\oplus m}\ , \tag{4.2.26}$$

and $\mathtt{lin}_k^* : \mathfrak{G}^{\rm op} \to \mathtt{Vect}_k$ denotes its composition with linear duality. Thus, for X n-connected, the E^2-terms of the spectral sequence (4.2.18) can be identified as

$$\begin{aligned} E^2_{00} &\cong k\ , \quad E^2_{p,0} \cong \mathrm{Tor}_p^{\mathfrak{G}}(\underline{k},\ \underline{\mathcal{H}}) \quad \text{for } p>0\ , \\ E^2_{pq} &= 0 \quad \text{for } 1\le q\le n-1\ ,\ p\ge 0\ , \\ E^2_{pq} &\cong \mathrm{Tor}_p^{\mathfrak{G}}(\mathtt{lin}_k^*,\ \underline{\mathcal{H}})\otimes \mathrm{H}_{q+1}(X;k) \quad \text{for } n\le q\le n-1\ ,\ p\ge 0\ . \end{aligned} \tag{4.2.27}$$

By Lemma 4.2.16, the right $\mathfrak{G}$-module $\underline{k} = k\langle 0\rangle$ is projective. Hence, $E^2_{p,0} = 0$ for $p>0$. On the other hand, by Proposition 6.3.6 (see Section 6.3 below), $\mathtt{lin}_k^*\otimes_{\mathfrak{G}}\underline{\mathcal{H}} \cong \mathfrak{g}^*$, while $\mathrm{Tor}_p^{\mathfrak{G}}(\mathtt{lin}_k^*,\ \underline{\mathcal{H}}) = 0$ for $p>0$. Hence, for $n\le q\le 2n-1$, we have

$$E^2_{0,q} = \mathfrak{g}^*\otimes \mathrm{H}_{q+1}(X;k) \cong \mathrm{H}_{q+1}(X;\mathfrak{g}^*)\ , \qquad E^2_{pq} = 0 \quad \text{for } p>0\ . \tag{4.2.28}$$

The vanishing of E^2_{pq} for all $p>0$ in the range $0\le q\le 2n-1$ shows that the spectral sequence (4.2.18) collapses on the q-axis for these values of q. Thus, we have $\mathrm{HR}_q(X,\underline{\mathcal{H}}) \cong E^2_{0,q}$ for $0\le q\le 2n-1$. By (4.2.27) and (4.2.28), these are the desired isomorphisms (4.2.25). □

Remark 4.2.29. For an arbitrary (pointed connected) topological space X, the 5-term exact sequence associated to the spectral sequence (4.2.18) reads

$$\mathrm{HR}_2(X,\mathcal{H}) \to \mathrm{Tor}_2^{\mathfrak{G}}(k[\pi_1(X)],\ \underline{\mathcal{H}}) \to \underline{\mathrm{H}}_1(\Omega X;k)\otimes_{\mathfrak{G}}\underline{\mathcal{H}} \to \mathrm{HR}_1(X,\mathcal{H}) \to \mathrm{Tor}_1^{\mathfrak{G}}(k[\pi_1(X)],\ \underline{\mathcal{H}}) \to 0$$

If the fundamental group $\pi_1(X)$ is free, then by Lemma 4.2.16, the corresponding right $\mathfrak{G}$-module is flat, and hence in that case, we have an isomorphism

$$\mathrm{HR}_1(X,\mathcal{H}) \cong \underline{\mathrm{H}}_1(\Omega X;k)\otimes_{\mathfrak{G}}\underline{\mathcal{H}}\ .$$

It would be interesting to give an explicit interpretation of $\mathrm{HR}_1(X,\mathcal{H})$ for an arbitrary X similar to (4.2.25) in the case of simply-connected spaces.

4.3 Realization of representation homology

In this section, we give another construction of representation homology that does not use the Kan equivalence (4.1.2). Our starting point is a general categorical principle which asserts that any left adjoint functor on the category of simplicial sets with values in a (cocomplete) category $\mathscr{C}$ arises from a cosimplicial object in $\mathscr{C}$. Specifically, a cosimplicial object $F : \Delta \to \mathscr{C}$ gives rise to the simplicial adjunction

$$- \otimes_{\Delta} F : \ \mathtt{sSet} \rightleftarrows \mathscr{C} \ : \mathrm{Hom}_{\mathscr{C}}(F, -)\,,$$

where $(- \otimes_{\Delta} F)$ denotes the left Kan extension of F along the Yoneda embedding $Y : \Delta \to \mathtt{sSet}$, and $\mathrm{Hom}_{\mathscr{C}}(F, -)$ is the functor assigning to $A \in \mathrm{Ob}(\mathscr{C})$ the simplicial set $\{\mathrm{Hom}_{\mathscr{C}}(F([n]), A)\}_{n\geq 0}$. This gives an equivalence between the category $\mathscr{C}^{\Delta}$ of cosimplicial objects in $\mathscr{C}$ and the category of simplicial adjunctions with values in $\mathscr{C}$ (see, e.g., [79, Prop. 3.1.5]).

The fundamental example is the cosimplicial space $\Delta^* \in \mathtt{Top}^{\Delta}$ defined by the geometric simplices $\{\Delta^n\}_{n\geq 0}$. Under the above equivalence, it corresponds to the classical adjunction between simplicial sets and topological spaces:

$$|-| : \ \mathtt{sSet} \rightleftarrows \mathtt{Top} \ : \mathrm{Hom}_{\mathtt{Top}}(\Delta^*, -)\,,$$

where $|-| = - \otimes_{\Delta} \Delta^*$ is the geometric realization functor defined in Section 2.2. In general, it is therefore natural to think of functors of the form $(- \otimes_{\Delta} F)$: $\mathtt{sSet} \to \mathscr{C}$ as *realization functors* of simplicial sets in $\mathscr{C}$.

Now, let us take an affine algebraic group G and consider its classifying space BG. This is naturally a simplicial affine k-scheme, and its coordinate ring $\mathcal{O}(BG) : \Delta \to \mathtt{CommAlg}_k$ is a cosimplicial commutative k-algebra. The next proposition shows that the 'realization' functor corresponding to $\mathcal{O}(BG)$ is just the classical representation functor $(-)_G$ (see Section 4.2).

Proposition 4.3.1. *For any $X \in \mathtt{sSet}_0$, there is a natural isomorphism of commutative algebras*

$$X \otimes_\Delta \mathcal{O}(BG) \cong \pi_1(X)_G = \mathcal{O}[\mathrm{Rep}_G(\pi_1(X))]$$

In particular, if Γ is a discrete group, then $\mathrm{B}\Gamma \otimes_\Delta \mathcal{O}(BG) \cong \Gamma_G$.

Proof. For any $A \in \mathtt{CommAlg}_k$, we have canonical isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathtt{CommAlg}_k}(X \otimes_\Delta \mathcal{O}(BG), A) &\cong \mathrm{Hom}_{\mathtt{sSet}}(X, \mathrm{Hom}_{\mathtt{CommAlg}_k}(\mathcal{O}(BG), A)) \\
&\cong \mathrm{Hom}_{\mathtt{sSet}_0}(X, BG(A)) \\
&\cong \mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}X, G(A)) \\
&\cong \mathrm{Hom}_{\mathtt{Gr}}(\pi_1(X), G(A)) \\
&\cong \mathrm{Hom}_{\mathtt{CommAlg}_k}(\pi_1(X)_G, A)\ .
\end{aligned}$$

Here, the third and fourth isomorphisms follow from the fact that $G(A)$ is a *discrete* simplicial group, so that $BG(A) = \overline{W}G(A)$ and $\mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}X, G(A)) = \mathrm{Hom}_{\mathtt{Gr}}(\pi_0(\mathbb{G}X), G(A))$. The desired proposition follows now from Yoneda Lemma. $\square$

Proposition 4.3.1 suggests the possibility of defining representation homology of spaces in terms of the (non-abelian) derived tensor product $\otimes^{\boldsymbol{L}}_\Delta$. We briefly outline this construction and refer the reader to Part II for more details and proofs. We will need some technical results from abstract homotopy theory for which we will refer to [77] and [140].

From now on, we let $\mathscr{C} = s\mathtt{CommAlg}_k$ denote the category of simplicial commutative k-algebras. This is a simplicial model category, tensored and cotensored over $\mathtt{sSet}$. Given two functors $X : \Delta^{\mathrm{op}} \to \mathtt{sSet}$ and $F : \Delta \to \mathscr{C}$, we recall that the functor tensor product $X \otimes_\Delta F$ is defined by the coequilizer (*cf.* [140, (4.1.1)])

$$X \otimes_\Delta F := \mathrm{coeq}\left(\coprod_{f:[n]\to[m]} X([m]) \otimes F([n]) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{[n]\in\Delta} X([n]) \otimes F([n]) \right) .$$

The assignment $(X, F) \mapsto X \otimes_\Delta F$ defines a bifunctor $(- \otimes_\Delta -) : \mathtt{sSet}^{\Delta^{\mathrm{op}}} \times \mathscr{C}^\Delta \to \mathscr{C}$, which we would like to replace by a homotopy invariant derived functor. To this end, we equip the categories $\mathtt{sSet}^{\Delta^{\mathrm{op}}}$ and $\mathscr{C}^\Delta$ with Reedy model structures, where the weak equivalences are given by the pointwise weak equivalences of diagrams in the underlying model categories $\mathtt{sSet}$ and $\mathscr{C}$. Relative to the Reedy product model structure on $\mathtt{sSet}^{\Delta^{\mathrm{op}}} \times \mathscr{C}^\Delta$ and the natural (simplicial) model structure on $\mathscr{C}$, the functor tensor product $(- \otimes_\Delta -)$ then becomes a Quillen bifunctor (see [77, Theorem 19.7.2 and Corollary 19.7.4]). This means, in particular, that $(- \otimes_\Delta -)$ is a left Quillen functor with respect to each of its arguments, provided the other argument is chosen to be a Reedy cofibrant object. Moreover, there is a well-defined total left derived functor

$$- \overset{\boldsymbol{L}}{\underset{\Delta}{\otimes}} - : \mathrm{Ho}\big(\mathtt{sSet}^{\Delta^{\mathrm{op}}}\big) \times \mathrm{Ho}\big(\mathscr{C}^\Delta\big) \to \mathrm{Ho}(\mathscr{C}) .$$

Now, it is known (see [77, Corollary 15.8.8]) that the category $\mathtt{sSet}^{\Delta^{\mathrm{op}}}$ is Reedy cofibrant, *i.e.* every simplicial object in $\mathtt{sSet}$ is cofibrant with respect to the Reedy model structure on $\mathtt{sSet}^{\Delta^{\mathrm{op}}}$. Hence, to compute the derived tensor product $X \otimes^{\boldsymbol{L}}_\Delta F$ it suffices only to replace the diagram F by its Reedy cofibrant model in $\mathscr{C}^\Delta$. If $F : \Delta \to \mathscr{C}$ is a pointwise cofibrant diagram (with respect to the natural model structure) in $\mathscr{C}$, then F has a *canonical* Reedy cofibrant model in $\mathscr{C}^\Delta$ given by its bar construction (see [140, Sec. 4.2]).

Theorem 4.3.2. *[Same as Theorem 12.2.3] For any $X \in \mathtt{sSet}_0$, there is a natural* *isomorphism in Ho($\mathscr{C}$) :*

$$\mathrm{DRep}_G(\mathbb{G}X) \;\cong\; X \overset{\boldsymbol{L}}{\underset{\Delta}{\otimes}} \mathcal{O}(BG)\,. \tag{4.3.3}$$

Sketch of proof. The cosimplicial commutative algebra $\mathcal{O}(BG) \in \mathscr{C}^{\Delta}$ is given (in cosimplicial degree m) by the commutative algebra

$$\mathcal{O}(BG)^m = \mathcal{O}(G)^{\otimes(m+1)}$$

This cosimplicial commutative algebra has a canonical resolution by a cosimplicial simplicial commutative algebra, denoted $\mathcal{O}(\overline{W}RG) \in \mathscr{C}^{\Delta}$, which is given (in cosimplicial degree m) by the simplicial commutative algebra

$$\mathcal{O}(\overline{W}RG)^m = (\mathcal{O}(G) \otimes \Delta[m]) \,\otimes\, (\mathcal{O}(G) \otimes \Delta[m-1]) \,\otimes\, \ldots \,\otimes\, (\mathcal{O}(G) \otimes \Delta[0])$$

We refer the reader to Part II for a precise definition of $\mathcal{O}(\overline{W}RG)$. One can show that there is an isomorphism of simplicial commutative algebras

$$(\mathbb{G}(X))_G \;\cong\; X \otimes_{\Delta} \mathcal{O}(\overline{W}RG)\,. \tag{4.3.4}$$

Although the cosimplicial simplicial commutative algebra $\mathcal{O}(\overline{W}RG) \in \mathscr{C}^{\Delta}$ is not Reedy cofibrant, using (4.3.4), one can show that it is Reedy smooth in the sense of Section 11, Definition 11.3.6. Theorem 4.3.2 is therefore a consequence of Proposition 11.3.13. $\square$

Remark 1. Formula (4.3.3) may be thought of as an analogue of the Loday construction for higher Hochschild homology (see Section 3.2). Indeed, the Loday construction can be interpreted as follows. Given a commutative algebra $A \in \mathrm{Ob}(\mathscr{C})$, consider the functor $\underline{A} : \Delta \xrightarrow{Y} \mathtt{sSet} \xrightarrow{-\otimes A} \mathscr{C}$, defining a cosimplicial object in $\mathscr{C}$. Then, it follows from Lemma 3.2.2 that $\mathrm{HH}_*(X, A) \;\cong\; \pi_*[X \overset{\boldsymbol{L}}{\underset{\Delta}{\otimes}} \underline{A}]$

for any smooth commutative algebra A. The results of the next section suggest that, for any space X, there is a homotopy equivalence

$$\Sigma(X_+) \overset{\boldsymbol{L}}{\underset{\Delta}{\otimes}} \mathcal{O}(BG) \simeq X \overset{\boldsymbol{L}}{\underset{\Delta}{\otimes}} \mathcal{O}(G) .$$

Remark 2. Instead of resolving $\mathcal{O}(BG)$ by a cosimplicial simplicial commutative algebra, one could resolve it by a cosimplicial commutative DG algebra $\mathcal{O}(RBG) \in (\mathtt{DGCA}_k^+)^\Delta$. The resulting tensor product $X \otimes_\Delta \mathcal{O}(RBG) \in \mathtt{DGCA}_k^+$ will then correspond to the derived representation scheme under the Quillen equivalence $\mathrm{Ho}(\mathtt{sCommAlg}_k) \simeq \mathrm{Ho}(\mathtt{DGCA}_k^+)$ (see Theorem 10.3.2).

As in the case of simplicial commutative algebras, it suffices to require the resolution $\mathcal{O}(RBG)$ to be Reedy smooth in $(\mathtt{DGCA}_k^+)^\Delta$, meaning that all the latching maps are required to be smooth extensions of commutative DG algebras in an appropriate sense (see Definition 11.3.6). The simplicial DG scheme X corresponding to a Reedy smooth cosimplicial commutative DG algebra $\mathcal{O}(X)$ is called *injective* in [96]. Thus, the proof of Theorem 4.3.2 shows that the homotopy type of the derived representation scheme coincides with the homotopy type of the derived space $\mathbf{R}\mathrm{Loc}(X)$ of local systems defined in [96].

CHAPTER 5

RELATION BETWEEN REPRESENTATION HOMOLOGY AND HOCHSCHILD HOMOLOGY

In Section 4.2, we defined representation homology by analogy with Hochschild homology, using Kan's simplicial loop group construction. In this section, we establish a direct relation between these two homology theories using another classical construction in simplicial homotopy theory due to J. Milnor [114].

5.1 Main theorems

We begin by recalling a standard simplicial model for a (reduced) suspension ΣX of a space X. The *suspension functor* on pointed simplicial sets is defined by

$$\Sigma : \mathtt{sSet}_* \to \mathtt{sSet}_0 , \quad X \mapsto C(X)/X ,$$

where $C(X) \in \mathtt{sSet}_*$ is the reduced cone over X. For a pointed simplicial set $X = \{X_n\}_{n\geq 0}$, the set of n-simplices in $C(X)$ is given by

$$C(X)_n := \{(x, m) : x \in X_{n-m} , 0 \leq m \leq n\} ,$$

with all $(*, m)$ being identified to $*$. The face and degeneracy maps in $C(X)$ are defined by

$$d_i : C(X)_n \to C(X)_{n-1} , \qquad (x, m) \mapsto \begin{cases} (x, m-1) & \text{if } 0 \leq i < m \\ (d^X_{i-m}(x), m) & \text{if } m \leq i \leq n \end{cases}$$

$$s_j : C(X)_n \to C(X)_{n+1} , \qquad (x, m) \mapsto \begin{cases} (x, m+1) & \text{if } 0 \leq j < m \\ (s^X_{j-m}(x), m) & \text{if } m \leq j \leq n \end{cases}$$

where $d_1(x,1) = *$ for all $x \in X_0$.

The embedding $X \hookrightarrow C(X)$ is given by $x \mapsto (x,0)$, and ΣX is defined to be the corresponding quotient set. Note that, unlike $C(X)$, the simplicial set ΣX is reduced, since $(x,0) = *$ in ΣX for all $x \in X$ (in particular, we have $C(X)_0 = \{(x,0) : x \in X_0\} \sim \{*\}$). Now, for any pointed simplicial set X, there is a homotopy equivalence $|\Sigma X| \simeq \Sigma|X|$, where $\Sigma|X|$ is reduced suspension of the geometric realization of X in the usual topological sense.

The next two theorems constitute the main result of this section.

Theorem 5.1.1. *For any commutative Hopf algebra $\mathcal{H}$ and any pointed simplicial set X, there is a natural isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(\Sigma X, \mathcal{H}) \cong \mathrm{HH}_*(X, \mathcal{H}; k)\,.$$

To state the next theorem, we recall that there is a natural way to make an arbitrary simplicial set pointed by adding to it a disjoint basepoint. To be precise, the forgetful functor $\mathtt{sSet}_* \to \mathtt{sSet}$ has a left adjoint $(-)_+ : \mathtt{sSet} \to \mathtt{sSet}_*$ obtained by extending to simplicial sets the obvious functor $X \mapsto X \sqcup \{*\}$ on the category of sets. Explicitly, if $\{X_n\}_{n\geq 0}$ is a simplicial set, then $(X_+)_n = X_n \sqcup \{*\}$ for all n, and the face and degeneracy maps of X_+ are the (unique) basepoint-preserving extensions of the corresponding maps of X. Being a left adjoint, the functor $(-)_+$ commutes with colimits; in particular, we have

$$|X_+| \cong |X|_+\,,$$

where $|X|_+$ is the space obtained from $|X|$ by adjoining a basepoint.

Theorem 5.1.2. *For any commutative Hopf algebra $\mathcal{H}$ and any simplicial set X, there is an isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \cong \mathrm{HH}_*(X, \mathcal{H})\,.$$

The proofs of Theorem 5.1.1 and Theorem 5.1.2 are based on a classical simplicial group model of the spaces $\Omega\Sigma X$, which we now briefly review.

5.2 Milnor's FK-construction

For a pointed simplicial set $K \in \mathtt{sSet}_*$, we define $FK := \mathbb{G}\Sigma K$. Then, by Kan's Theorem 4.1.4, there is a homotopy equivalence of spaces

$$|FK| \simeq \Omega\Sigma|K|\,.$$

The following observation is due to J. Milnor [114] (see also [67, Theorem V.6.15]).

Lemma 5.2.1 (Milnor). *For any $K \in \mathtt{sSet}_*$, FK is a semi-free simplicial group generated by the simplicial set K with basepoint identified with 1, i.e.*

$$FK_n = (\mathbb{G}\Sigma K)_n \cong \langle K_n\rangle/\langle s_0^n(*) = 1\rangle \cong \langle K_n\backslash s_0^n(*)\rangle\,.$$

The face and degeneracy maps are induced by the face and degeneracy maps of K.

Proof. By definition of the reduced suspension, we have $(x,0) = *$ for all $x \in K$ and $s_0(x,m) = (x,m{+}1)$ for all $m > 0$. Hence, $(\Sigma K)_{n+1}/s_0(\Sigma K_n) = \{(x,1)\,|\,x \in K_n\}$, with $(*,1)$ being the basepoint. It follows that

$$(\mathbb{G}\Sigma K)_n = \langle(\Sigma K)_{n+1}/s_0(\Sigma K_n)\rangle \cong \langle K_n\rangle/(* = 1)\,.$$

To calculate the face and degeneracy maps, we recall from Section 4.1 that

$$d_0^{\mathbb{G}\Sigma K}(x,1) = d_1(x,1)\,d_0(x,1)^{-1} = (d_0x,1)\,(x,0)^{-1} = (d_0x,1)\,,$$

and $d_i^{\mathbb{G}\Sigma K}(x,1) = d_{i+1}(x,1) = (d_ix,1)$ for $i > 0$. Similarly, $s_j^{\mathbb{G}\Sigma K}(x,1) = s_{j+1}(x,1) = (s_jx,1)$ for all $j \geq 0$. This proves the desired lemma. □

Proofs of Theorems 5.1.1 and Theorem 5.1.2

Recall that, for a commutative Hopf algebra $\mathcal{H}$, we denote by $\underline{\mathcal{H}}$ the functor $\mathtt{FGr} \to \mathtt{CommAlg}_k$ on the category of based free groups obtained from the $\mathfrak{G}$-module $\langle n \rangle \mapsto \mathcal{H}^{\otimes n}$ by taking its left Kan extension along the inclusion $\mathfrak{G} \hookrightarrow \mathtt{FGr}$ (see Section 4.2).

Proposition 5.2.2. *There is an isomorphism of functors from* $\mathtt{sSet}$ *to* $\mathtt{sCommAlg}_k$ *:*

$$\underline{\mathcal{H}} \circ \mathbb{G} \circ \Sigma \circ (-)_+ \cong (- \otimes \mathcal{H})\,,$$

where $\mathcal{H}$ in the right-hand side is regarded as a commutative k-algebra.

Proof. By Lemma 5.2.1, for any simplicial set $X = \{X_n\}_{n\geq 0}$, there are natural isomorphisms of groups $[\mathbb{G}\Sigma(X_+)]_n \cong \langle X_n \rangle\,,\ n \geq 0\,$, with structure maps on $\mathbb{G}\Sigma(X_+)$ being compatible with those of X. By applying the functor $\mathcal{H}$, we thus get isomorphisms of simplicial commutative algebras

$$\underline{\mathcal{H}}([\mathbb{G}\Sigma(X_+)]_*) \cong \underline{\mathcal{H}}[\langle X_* \rangle] \cong X_* \otimes \mathcal{H}\ ,$$

which are obviously functorial in X. This proves the proposition. $\square$

Theorem 5.1.2 is an immediate consequence of the above proposition. To prove Theorem 5.1.1, we first note that, although the unreduced cone on a space X coincides with the reduced cone on X_+, the corresponding suspensions differ. Instead, for any pointed space X, there is a homotopy equivalence (see [112, p. 106])

$$\Sigma(X_+) \simeq \Sigma X \vee \mathbb{S}^1\,. \tag{5.2.3}$$

From this we can deduce the following

Lemma 5.2.4. *For a pointed topological space X, there is a natural isomorphism*

$$\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \;\cong\; \mathrm{HR}_*(\Sigma X, \mathcal{H}) \otimes \mathcal{H}.$$

Proof. Applying Corollary 4.2.14 to (5.2.3), we have $\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \cong \mathrm{HR}_*(\Sigma X, \mathcal{H}) \otimes \mathrm{HR}_*(\mathbb{S}^1, \mathcal{H})$. Now, since $\mathbb{S}^1 \cong \Sigma(\mathrm{pt}_+)$, Theorem 5.1.2 implies $\mathrm{HR}_*(\mathbb{S}^1, \mathcal{H}) \cong \mathrm{HH}_*(\mathrm{pt}, \mathcal{H}) \cong \mathcal{H}$, where $\mathcal{H}$ is concentrated in degree 0. It follows that $\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \cong \mathrm{HR}_*(\Sigma X, \mathcal{H}) \otimes \mathcal{H}$ as desired. □

Lemma 5.2.4 shows that $\mathrm{HR}_*(\Sigma X, \mathcal{H}) \cong \mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \otimes_{\mathcal{H}} k$. Combining this last isomorphism with that of Theorem 5.1.2, we now conclude

$$\mathrm{HR}_*(\Sigma X, \mathcal{H}) \;\cong\; \mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \otimes_{\mathcal{H}} k \;\cong\; \mathrm{HH}_*(X, \mathcal{H}) \otimes_{\mathcal{H}} k \;\cong\; \mathrm{HH}_*(X, \mathcal{H}; k)\ .$$

This proves Theorem 5.1.1.

5.3 Examples

We conclude this section with a few simple examples illustrating the use of Theorems 5.1.1 and 5.1.2. More examples will be given in the next two sections. In what follows, G denotes an arbitrary affine algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$ stands for its Lie algebra.

Spheres

The representation homology of the circle $\mathbb{S}^1$ is given by $\mathrm{HR}_0(\mathbb{S}^1, G) \cong \mathcal{O}(G)$ and $\mathrm{HR}_i(\mathbb{S}^1, G) = 0$ for $i > 0$. This follows, for example, from Lemma 4.2.16

and Corollary 4.2.19 (since $\mathbb{S}^1 \cong B\mathbb{Z}$). Now, for higher dimensional spheres, we have

Proposition 5.3.1. $\mathrm{HR}_*(\mathbb{S}^n, G) \cong \mathrm{Sym}_k(\mathfrak{g}^*[n-1])$ *for all* $n \geq 2$.

Proof. Note that $\mathbb{S}^n \simeq \Sigma\,\mathbb{S}^{n-1}$ for all $n \geq 2$. By Theorem 5.1.1, we conclude

$$\mathrm{HR}_*(\mathbb{S}^n, G) \cong \mathrm{HH}_*(\mathbb{S}^{n-1}, \mathcal{O}(G); k) \cong \mathrm{Sym}_{\mathcal{O}(G)}(\Omega^1(G)[n-1]) \otimes_{\mathcal{O}(G)} k \cong \mathrm{Sym}_k(\mathfrak{g}^*[n-1])\,,$$

where the second isomorphism follows from [133, Section 5.5]. $\square$

Suspensions

We now generalize the previous example to arbitrary suspensions.

Proposition 5.3.2. *Let ΣX be the suspension of a pointed connected space X of finite type. Then*

$$\mathrm{HR}_*(\Sigma X, G) \cong \mathrm{Sym}_k\Bigg(\bigoplus_{n\geq 1} \mathrm{H}_n(X; \mathfrak{g}^*)[n]\Bigg)\,.$$

Proof. It is known (see [58, Theorem 24.5]) that ΣX is rationally homotopy equivalent to a bouquet of spheres: $\Sigma X \simeq_{\mathbb{Q}} \bigvee_{i\in I} \mathbb{S}^{n_i}$, where each $\mathbb{S}^{n_i}$ have dimension $n_i \geq 2$. By Proposition 4.2.23, it thus suffices to compute $\mathrm{HR}_*(S, G)$ for $S := \bigvee_{i\in I} \mathbb{S}^{n_i}$. Note that the reduced homology $\overline{\mathrm{H}}_*(S; k)$ of S is isomorphic to $\oplus_{i\in I}\, k \cdot v_i$ with trivial coproduct, where v_i is a basis element of homological degree $\deg(v_i) = n_i$. Now, by Corollary 4.2.14 and Proposition 5.3.1, we have

$$\begin{aligned}
\mathrm{HR}_*(\Sigma X, G) &\cong \mathrm{HR}_*(S, G) \cong \bigotimes_{i\in I} \mathrm{HR}_*(\mathbb{S}^{n_i}, G) \cong \bigotimes_{i\in I} \mathrm{Sym}_k(\mathfrak{g}^*[n_i - 1]) \\
&\cong \mathrm{Sym}_k\Bigg(\bigoplus_{i\in I} \mathfrak{g}^*[n_i - 1]\Bigg) \cong \mathrm{Sym}_k\Bigg(\bigoplus_{n\geq 2} \mathfrak{g}^* \otimes \mathrm{H}_n(\Sigma X; k)[n-1]\Bigg) \\
&\cong \mathrm{Sym}_k\Bigg(\bigoplus_{n\geq 1} \mathfrak{g}^* \otimes \mathrm{H}_n(X; k)[n]\Bigg) \cong \mathrm{Sym}_k\Bigg(\bigoplus_{n\geq 1} \mathrm{H}_n(X; \mathfrak{g}^*)[n]\Bigg)\,,
\end{aligned}$$

where the last isomorphism follows from Universal Coefficient Theorem. This proves Proposition 5.3.2. $\square$

Remark 5.3.3. Proposition 5.3.2 actually holds at the chain level: namely, the derived representation schemes of the suspensions ΣX of connected spaces are formal, *i.e.* DG algebra models representing these derived schemes are quasi-isomorphic to their homology.

As a consequence of Theorem 5.1.2, Lemma 5.2.4 and Proposition 5.3.2, we have the following general formula for the higher Hochschild homology with coefficients in commutative Hopf algebras.

Proposition 5.3.4. *For any pointed connected topological space X of finite type,*

$$\mathrm{HH}_*(X, \mathcal{O}(G)) \;\cong\; \mathrm{Sym}_{\mathcal{O}(G)}\Bigg(\bigoplus_{n\geq 1} \mathrm{H}_n(X;k)\otimes \Omega^1(G)[n]\Bigg)\,. \tag{5.3.5}$$

We remark that the isomorphism (5.3.5) is a refinement of Pirashvili's generalization of the classical HKR Theorem which (in our notation) asserts that $\mathrm{HH}_*(X,A) \;\cong\; \underline{\mathrm{H}}_*(X;k) \otimes_{\mathfrak{F}} \underline{A}$ for any smooth commutative algebra A (*cf.* [133, Theorem 4.6]). Since both sides of (5.3.5) make sense with $\mathcal{O}(G)$ replaced by any smooth commutative algebra A, it is natural ask when

$$\mathrm{HH}_*(X,A) \;\cong\; \mathrm{Sym}_A\Bigg(\bigoplus_{n\geq 1} \mathrm{H}_n(X;k)\otimes \Omega^1(A)[n]\Bigg) \quad \text{for any connected space } X\,? \tag{5.3.6}$$

Note that, by Proposition 5.3.4, (5.3.6) holds for any polynomial algebra $A = k[x_1,\ldots,x_n]$, $n\geq 1$.

Remark 5.3.7. In recent years, there have been a number of interesting *topological* generalizations of higher Hochschild homology, such as factorization homology (see, e.g., [7, 74, 75]) and higher topological Hochschild homology (see [31]).

It is natural to expect that representation homology admits similar topological versions, and the relation between higher Hochschild homology and representation homology established in this section extends to this more refined topological context.

CHAPTER 6

REPRESENTATION HOMOLOGY OF SIMPLY-CONNECTED SPACES

If X is a simply-connected topoloical space of finite type, the rational homotopy type can be described by a differential graded Lie algebra $\mathcal{L}_X$ called a Lie model of X. In this section, we address the natural question: What is the relation between the representation homology of X and that of the Lie model $\mathcal{L}_X$? Our main theorem, which we call Comparison Theorem, asserts that the two homology theories are isomorphic.

6.1 Representation homology of Lie algebras

We begin by reviewing the definition of derived representation schemes of Lie algebras and associated character maps, which we call the Drinfeld traces. For details and proofs, we refer the reader to [24].

Representation functor

Let $\mathfrak{g}$ be a fixed finite-dimensional Lie algebra over k. Given an (arbitrary) Lie algebra $\mathfrak{a} \in \mathtt{Lie\,Alg}_k$, we are interested in classifying the representations of $\mathfrak{a}$ in $\mathfrak{g}$. The corresponding moduli scheme $\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a})$ is defined by its functor of points:

$$\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a}) : \mathtt{CommAlg}_k \to \mathtt{Set}\ , \quad B \mapsto \mathrm{Hom}_{\mathtt{Lie}}(\mathfrak{a},\, \mathfrak{g}(B))$$

assigning to a commutative k-algebra B the set of families of representations of $\mathfrak{a}$ in $\mathfrak{g}$ parametrized by the k-scheme $\mathrm{Spec}(B)$. The functor $\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a})$ is represented by a commutative algebra $\mathfrak{a}_{\mathfrak{g}}$, which has the following canonical presentation (*cf.* [24, Prop. 6.1]):

$$\mathfrak{a}_{\mathfrak{g}} \;=\; \frac{\mathrm{Sym}_k(\mathfrak{a}\otimes\mathfrak{g}^*)}{\langle\!\langle\,(x\otimes\xi_1^*)\cdot(y\otimes\xi_2^*)-(y\otimes\xi_1^*)\cdot(x\otimes\xi_2^*)-[x,y]\otimes\xi^*\,\rangle\!\rangle}\;, \tag{6.1.1}$$

where $\mathfrak{g}^*$ is the vector space dual to $\mathfrak{g}$ and $\xi^*\mapsto\xi_1^*\wedge\xi_2^*$ is the linear map $\mathfrak{g}^*\to\wedge^2\mathfrak{g}^*$ dual to the Lie bracket on $\mathfrak{g}$. The universal representation $\varrho_{\mathfrak{g}}:\mathfrak{a}\to\mathfrak{g}(\mathfrak{a}_{\mathfrak{g}})$ is given by the natural map

$$\mathfrak{a}\to\mathfrak{a}\otimes\mathfrak{g}^*\otimes\mathfrak{g}\;\hookrightarrow\;\mathrm{Sym}_k(\mathfrak{a}\otimes\mathfrak{g}^*)\otimes\mathfrak{g}\;\twoheadrightarrow\;\mathfrak{a}_{\mathfrak{g}}\otimes\mathfrak{g}=\mathfrak{g}(\mathfrak{a}_{\mathfrak{g}})\;,\quad x\mapsto\sum_i\,[x\otimes\xi_i^*]\otimes\xi_i\;, \tag{6.1.2}$$

where $\{\xi_i\}$ and $\{\xi_i^*\}$ are dual bases in $\mathfrak{g}$ and $\mathfrak{g}^*$. The algebra $\mathfrak{a}_{\mathfrak{g}}$ has a canonical augmentation $\varepsilon:\mathfrak{a}_{\mathfrak{g}}\to k$ induced by the zero map $\mathfrak{a}\otimes\mathfrak{g}^*\to 0$. Thus the assignment $\mathfrak{a}\mapsto\mathfrak{a}_{\mathfrak{g}}$ defines a functor with values in the category of augmented commutative algebras:

$$(\,-\,)_{\mathfrak{g}}\;:\;\mathtt{Lie\,Alg}_k\to\mathtt{CommAlg}_{k/k}\;. \tag{6.1.3}$$

We call (6.1.3) the *representation functor* in $\mathfrak{g}$. Geometrically, one can think of $(\mathfrak{a}_{\mathfrak{g}},\varepsilon)$ as a coordinate ring $k[\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a})]$ of the based affine scheme $\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a})$, with the basepoint corresponding to the trivial representation.

Next, let G be an affine algebraic group over k associated with the Lie algebra $\mathfrak{g}$. Observe that for any $\mathfrak{a}\in\mathtt{Lie\,Alg}_k$, G acts naturally on $\mathfrak{a}_{\mathfrak{g}}$ by automorphisms: this action is functorial in $\mathfrak{a}$. We write $(\,-\,)_{\mathfrak{g}}^G:\mathtt{Lie\,Alg}_k\to\mathtt{CommAlg}_{k/k}$ for the subfunctor of $(\,-\,)_{\mathfrak{g}}$ defined by taking the G-invariants:

$$\mathfrak{a}_{\mathfrak{g}}^G:=\{x\in\mathfrak{a}_{\mathfrak{g}}\;:\;g(x)=x\,,\,\forall\,g\in G\}\;.$$

The algebra $\mathfrak{a}_{\mathfrak{g}}^G$ represents the affine quotient scheme $\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a})/\!/G$ parametrizing the closed orbits of G in $\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a})$.

Derived functors

The functor (6.1.3) can be extended to the category of DG Lie algebras

$$(\,-\,)_{\mathfrak{g}}\;:\;\mathtt{DGLA}_k\to\mathtt{DGCA}_{k/k}\;,\quad\mathfrak{a}\mapsto\mathfrak{a}_{\mathfrak{g}}\;. \tag{6.1.4}$$

It is shown in [24] that, for a fixed $\mathfrak{a}\in\mathtt{DGLA}_k$, the corresponding commutative DG algebra $\mathfrak{a}_{\mathfrak{g}}$ represents an affine DG scheme parametrizing the DG Lie representations of $\mathfrak{a}$ in $\mathfrak{g}$. Now, although the functor (6.1.4) is not homotopy invariant (it does not preserve quasi-isomorphisms and hence does not descend to $\mathrm{Ho}(\mathtt{DGLA}_k)$), it is a left Quillen functor and hence has a well-behaved left derived functor (see [24, Theorem 6.4])

$$\boldsymbol{L}(\,-\,)_{\mathfrak{g}}\;:\;\mathrm{Ho}(\mathtt{DGLA}_k)\;\to\;\mathrm{Ho}(\mathtt{DGCA}_{k/k})\;.$$

For a fixed DG Lie algebra $\mathfrak{a}\in\mathtt{DGLA}_k$, we then define $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})\;:=\;\boldsymbol{L}(\mathfrak{a})_{\mathfrak{g}}\,$.

In a similar fashion, we construct the derived functor of the invariant functor $(\,-\,)_{\mathfrak{g}}^G$:

$$\boldsymbol{L}(\,-\,)_{\mathfrak{g}}^G\;:\;\mathrm{Ho}(\mathtt{DGLA}_k)\;\to\;\mathrm{Ho}(\mathtt{DGCA}_{k/k})\;,$$

and define $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G\;:=\;\boldsymbol{L}(\mathfrak{a})_{\mathfrak{g}}^G$. Note that, abusing notation, we write $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ and $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ for the commutative DG algebras in $\mathrm{Ho}(\mathtt{DGCA}_{k/k})$ instead of the eponymous derived schemes. To compute $\boldsymbol{L}(\,-\,)_{\mathfrak{g}}$ and $\boldsymbol{L}(\,-\,)_{\mathfrak{g}}^G$ we use the cofibrant resolutions in $\mathtt{DGLA}_k$, which, in practice, are given by semi-free DG Lie algebras. Thus, we have isomorphisms

$$\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})\cong(Q\mathfrak{a})_{\mathfrak{g}}\;,\quad\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G\cong(Q\mathfrak{a})^G\;,$$

where $Q\mathfrak{a}$ is a(ny) cofibrant resolution of $\mathfrak{a}$ in $\mathtt{DGLA}_k$.

Finally, for any DG Lie algebra $\mathfrak{a}$, we define the *representation homology* of $\mathfrak{a}$ in $\mathfrak{g}$ by

$$\mathrm{HR}_*(\mathfrak{a},\mathfrak{g}) \,:=\, \mathrm{H}_*[\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})]\ .$$

This is a graded commutative k-algebra, which depends on $\mathfrak{a}$ and $\mathfrak{g}$ (but not on the choice of resolution of $\mathfrak{a}$). If $\mathfrak{a} \in \mathtt{Lie\,Alg}_k$ is an ordinary Lie algebra, then there is an isomorphism of commutative algebras $\mathrm{H}_0(\mathfrak{a},\mathfrak{g}) \cong \mathfrak{a}_{\mathfrak{g}}$ which justifies the name 'derived representation scheme' for $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$. In addition, if G is reductive, we have

$$\mathrm{HR}_*[\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G] \,\cong\, \mathrm{HR}_*(\mathfrak{a},\mathfrak{g})^G.$$

Thus, the homology of $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ can be viewed as the invariant part of the representation homology of $\mathfrak{a}$.

Drinfeld traces

Our next goal is to describe certain natural maps with values in representation homology. These maps can be viewed as (derived) characters of finite-dimensional Lie representations. From now on, we assume that $\mathfrak{g}$ is a reductive Lie algebra over k. We let $I(\mathfrak{g}) \,:=\, \mathrm{Sym}(\mathfrak{g}^*)^G$ denote the space of invariant polynomials on $\mathfrak{g}$, and for each $d \geq 0$, we write $I^d(\mathfrak{g}) \subset I(\mathfrak{g})$ for the subspace of homogeneous polynomials of degree d.

Consider the symmetric ad-invariant k-multilinear forms on $\mathfrak{a}$ of degree $d \geq 1$. Every such form is induced from the universal one: $\mathfrak{a} \times \mathfrak{a} \times \ldots \times \mathfrak{a} \to \lambda^{(d)}(\mathfrak{a})$, which takes its values in the space $\lambda^{(d)}(\mathfrak{a})$ of coinvariants of the adjoint representation of $\mathfrak{a}$ in $\mathrm{Sym}^d(\mathfrak{a})$. The assignment $\mathfrak{a} \mapsto \lambda^{(d)}(\mathfrak{a})$ defines a (non-additive) functor on the category of Lie algebras that extends in a natural way to the category of DG Lie algebras:

$$\lambda^{(d)} : \mathtt{DGLA}_k \to \mathtt{Com}_k\ , \quad \mathfrak{a} \mapsto \mathrm{Sym}^d(\mathfrak{a})/[\mathfrak{a}, \mathrm{Sym}^d(\mathfrak{a})]\ . \tag{6.1.5}$$

It can be shown (see [24, Thm. 7.1]) that the functor $\lambda^{(d)}$ has a (left) derived functor

$$\boldsymbol{L}\lambda^{(d)} : \mathrm{Ho}(\mathtt{DGLA}_k) \to \mathscr{D}(k)\ , \tag{6.1.6}$$

which takes its values in the derived category $\mathscr{D}(k)$ of k-complexes. We write $\mathrm{HC}_*^{(d)}(\mathfrak{a})$ for the homology of $\boldsymbol{L}\lambda^{(d)}(\mathfrak{a})$ and call it the *Lie-Hodge homology* of $\mathfrak{a}$. This terminology is justified by the fact that there is a natural direct sum decomposition (see [24, Sec. 7.2] and [26, Sec. 1])

$$\mathrm{HC}_*(U\mathfrak{a}) \cong \bigoplus_{d=1}^{\infty} \mathrm{HC}_*^{(d)}(\mathfrak{a})\ , \tag{6.1.7}$$

that is a Koszul dual of the classical Hodge decomposition of cyclic homology of commutative algebras.

Now, observe that, for any commutative algebra B, there is a natural symmetric invariant d-linear form $\mathfrak{a}(B) \times \mathfrak{a}(B) \times \ldots \times \mathfrak{a}(B) \to \lambda^{(d)}(\mathfrak{a}) \otimes B$ on the current Lie algebra $\mathfrak{a}(B)$. Hence, by the universal property of $\lambda^{(d)}$, we have a canonical map

$$\lambda^{(d)}[\mathfrak{a}(B)] \to \lambda^{(d)}(\mathfrak{a}) \otimes B\ . \tag{6.1.8}$$

Applying $\lambda^{(d)}$ to the universal representation (6.1.2) and composing with (6.1.8), we define

$$\lambda^{(d)}(\mathfrak{a}) \to \lambda^{(d)}[\mathfrak{g}(\mathfrak{a}_{\mathfrak{g}})] \to \lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{a}_{\mathfrak{g}}\ . \tag{6.1.9}$$

On the other hand, for the Lie algebra $\mathfrak{g}$, we have a canonical (nondegenerate) pairing

$$I^d(\mathfrak{g}) \times \lambda^{(d)}(\mathfrak{g}) \to k \tag{6.1.10}$$

induced by the linear pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Replacing the Lie algebra $\mathfrak{a}$ in (6.1.9) by its cofibrant resolution $\mathcal{L} \overset{\sim}{\twoheadrightarrow} \mathfrak{a}$ and using (6.1.10), we define the morphism of complexes

$$I^d(\mathfrak{g}) \otimes \lambda^{(d)}(\mathcal{L}) \xrightarrow{(6.1.9)} I^d(\mathfrak{g}) \otimes \lambda^{(d)}(\mathfrak{g}) \otimes \mathcal{L}_{\mathfrak{g}} \xrightarrow{(6.1.10)} \mathcal{L}_{\mathfrak{g}} \ . \tag{6.1.11}$$

For a fixed polynomial $P \in I^d(\mathfrak{g})$, this morphism induces a map on homology $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) \ : \ \mathrm{HC}_*^{(d)}(\mathfrak{a}) \ \to \ \mathrm{HR}_*(\mathfrak{a}, \mathfrak{g})$, which we call the *Drinfeld trace* associated [1] to P. It is easy to check that the image of (6.1.11) is contained in the invariant subalgebra $\mathcal{L}_{\mathfrak{g}}^G$ of $\mathcal{L}_{\mathfrak{g}}$, hence the Drinfeld trace is actually a map

$$\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) : \ \mathrm{HC}_*^{(d)}(\mathfrak{a}) \ \to \ \mathrm{HR}_*(\mathfrak{a}, \mathfrak{g})^G \, . \tag{6.1.12}$$

6.2 Comparison Theorem

In this section for simplicity, we assume that $k = \mathbb{Q}$ to use directly results from Quillen's rational homotopy paper [136]. However, as explained in Remark 6.3, the results of this section extend to an arbitrary field of characteristic 0 by a universal coefficient argument.

Let X be a 1-connected topological space of finite rational type. Recall (*cf.* [58]) that one can associate to X a commutative cochain DG algebra $\mathcal{A}_X$, called a *Sullivan model* of X, and a connected (chain) DG Lie algebra $\mathcal{L}_X$, called a *Quillen model* of X. Each of these algebras is uniquely determined up to homotopy and

[1] We warn the reader that the map $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})$ does depend on the choice of P but we suppress this in our notation.

each encodes the rational homotopy type of X. The relation between them is given by a DG algebra quasi-isomorphism

$$\mathrm{C}^*(\mathcal{L}_X;\mathbb{Q}) \xrightarrow{\sim} \mathcal{A}_X\ , \tag{6.2.1}$$

where $\mathrm{C}^*(\mathcal{L}_X;\mathbb{Q})$ is the Chevalley-Eilenberg cochain complex of $\mathcal{L}_X$. The homology of $\mathcal{L}_X$ is the homotopy Lie algebra $L_X = \pi_*(\Omega X)_{\mathbb{Q}}$, while the cohomology of $\mathcal{A}_X$ is the rational cohomology algebra $\mathrm{H}^*(X;\mathbb{Q})$ of X. Among Quillen models of X, there is a *minimal* one given by a semi-free DG Lie algebra $(\mathbb{L}_X(V), d)$ generated by a graded $\mathbb{Q}$-vector space V with differential d satisfying $d(V) \subset [\,\mathbb{L}(V),\, \mathbb{L}(V)]$. Such a minimal model is determined uniquely up to (noncanonical) isomorphism. In particular, $V \cong \mathrm{H}_*(X;\mathbb{Q})[-1]$ (see [58], p. 326).

Now, given an algebraic group G, one can associate to a 1-connected space X two representation homologies: the representation homology $\mathrm{HR}_*(X, G)$ of X with coefficients in G (in sense of Section 4.2) and the representation homology $\mathrm{HR}_*(\mathcal{L}_X, \mathfrak{g})$ of a Lie model $\mathcal{L}_X$ of X with coefficients in the Lie algebra of G (in sense of Section 6.1). The following theorem, which is one of the main results of the present Part, shows that the two constructions agree.

Theorem 6.2.2. *For any affine algebraic group G with Lie algebra $\mathfrak{g}$, there is an isomorphism of graded commutative $\mathbb{Q}$-algebras*

$$\mathrm{HR}_*(X, G) \;\cong\; \mathrm{HR}_*(\mathcal{L}_X, \mathfrak{g}).$$

Theorem 6.2.2 can be restated in terms of the Sullivan model of X. Recall that for any commutative (DG) algebra $\mathcal{A}$, the tensor product $\mathfrak{g}(\mathcal{A}) := \mathfrak{g} \otimes \mathcal{A}$ has a natural DG Lie algebra structure with Lie bracket defined by $[\xi \otimes a\,,\, \eta \otimes b] = [\xi\,,\, \eta] \otimes ab$. In particular, if X is a pointed 1-connected topological space of finite rational type, its Sullivan model $\mathcal{A}_X$ is an augmented commutative DG algebra, and we can consider the DG Lie algebras $\mathfrak{g}(\mathcal{A}_X)$ and $\mathfrak{g}(\bar{\mathcal{A}}_X)$ both of which are cohomologically graded. We write $\mathrm{H}^{-*}(\mathfrak{g}(\bar{\mathcal{A}}_X);\mathbb{Q})$ and $\mathrm{H}^{-*}(\mathfrak{g}(\mathcal{A}_X),\mathfrak{g};\mathbb{Q})$ for the classical (absolute and relative) Chevalley-Eilenberg cohomologies of the Lie algebras $\mathfrak{g}(\bar{\mathcal{A}}_X)$ and $\mathfrak{g}(\mathcal{A}_X)$ equipped with *homological* grading. The following is a consequence of Theorem 6.2.2 and one of the main results of [24] (see *loc. cit.*, Theorem 6.5).

Theorem 6.2.3. *For any affine algebraic group G with Lie algebra $\mathfrak{g}$, there is an isomorphism of graded commutative algebras,*

$$\mathrm{HR}_*(X,G) \;\cong\; \mathrm{H}^{-*}(\mathfrak{g}(\bar{\mathcal{A}}_X);\mathbb{Q})\;. \tag{6.2.4}$$

Moreover, if G is reductive, there is an isomorphism of graded commutative algebras

$$\mathrm{HR}_*(X,G)^G \;\cong\; \mathrm{H}^{-*}(\mathfrak{g}(\mathcal{A}_X),\mathfrak{g};\mathbb{Q})\;.$$

Proof. Since the Sullivan model of X is uniquely determined up-to homotopy, it suffices to prove the desired theorem for a particular choice of Sullivan model of X. Let $\mathbb{L}_X := (\mathbb{L}_X(V), d)$ be the minimal Quillen model of X. Then, $\mathbb{L}_X$ is connected, i.e, concentrated in positive homological degree and finite dimensional in each homological degree. Hence, $C := \mathrm{C}_*(\mathbb{L}_X;\mathbb{Q})$ is 2-connected (i.e, its coaugmentation coideal is concentrated in degrees ≥ 2) and finite dimensional in each homological degree. The graded $\mathbb{Q}$-linear dual of C is $\mathcal{A}_X := \mathrm{C}^*(\mathbb{L}_X;\mathbb{Q})$, which is a Sullivan model of X. Moreover, C is Koszul dual to $\mathbb{L}_X$[2]. It follows from Theorem 6.2.2 and [24, Theorem 6.5 (b)] (also see *loc. cit.*, Theorem 6.3 and

[2]It is well known that if $\mathrm{char}(k) = 0$, there is a Quillen equivalence $\boldsymbol{\Omega}_{\mathtt{comm}} : \mathtt{DGCC}_{\mathbb{Q}/\mathbb{Q}} \rightleftarrows \mathtt{DGLA}_{\mathbb{Q}} : \mathrm{C}_*(-;\mathbb{Q})$ between the category of $\mathtt{DGCC}_{k/k}$ of (coaugmented, conilpotent) DG Lie coalgebras and the category $\mathtt{DGLA}_k$ of DG Lie algebras (see [83, Theorem 3.1 and 3.2]). Thus, there is a quasi-isomorphism of DG Lie algebras $\boldsymbol{\Omega}_{\mathtt{comm}}(C) \xrightarrow{\sim} \mathbb{L}_X$, which shows that C is Koszul dual to $\mathbb{L}_X$.

the subsequent remark) that

$$\mathrm{HR}_*(X, G) \;\cong\; \mathrm{HR}_*(\mathbb{L}_X, \mathfrak{g}) \;\cong\; \mathrm{H}^{-*}(\mathfrak{g}(\bar{\mathcal{A}}_X); \mathbb{Q})\ .$$

If, moreover, G is reductive, we have

$$\mathrm{HR}_*(X, G)^G \;\cong\; \mathrm{H}^{-*}(\mathfrak{g}(\bar{\mathcal{A}}_X); \mathbb{Q})^G \;\cong\; \mathrm{H}^{-*}(\mathfrak{g}(\bar{\mathcal{A}}_X); \mathbb{Q})^{\mathrm{ad}\,\mathfrak{g}} \;=\; \mathrm{H}^{-*}(\mathfrak{g}(\mathcal{A}_X), \mathfrak{g}; \mathbb{Q})\ .$$

The first isomorphism above follows from the fact that all (quasi-)isomorphisms in the proof of Theorem 6.2.2 are G-equivariant. Indeed, every G-action involved is induced by the G-action on the left $\mathfrak{G}$-module $\mathcal{O}(G)$ coming from the conjugation action of G on itself. This finishes the proof of the theorem. □

Before proving Theorem 6.2.2, we record one useful consequence that gives an explicit DG algebra model for the representation homology of X in terms of the minimal Quillen model of X.

Corollary 6.2.5. *Let $(\mathbb{L}_X(V), d)$ be the minimal Quillen model of X. Then, $(\mathbb{L}_X)_{\mathfrak{g}}$ is a canonical DG $\mathbb{Q}$-algebra whose homology is isomorphic to $\mathrm{HR}_*(X, G)$. Thus, as graded algebras,*

$$\mathrm{HR}_*(X, G) \;\cong\; \mathrm{H}_*[\mathrm{Sym}_k(\mathfrak{g}^* \otimes V), \partial]\,,$$

where the differential ∂ is given on generators by

$$\partial(\xi^* \otimes v) \;=\; \langle \xi^*, \varrho(dv)\rangle\ ,\ \ \forall\, \xi \in \mathfrak{g}^*,\ v \in V\ ,$$

where $\varrho\,:\, \mathbb{L}_X(V) \;\to\; \mathrm{Sym}(\mathfrak{g}^* \otimes V) \otimes \mathfrak{g}$ is the universal representation (6.1.2).

Proof. Since $\mathbb{L}_X$ is a semi-free (hence, cofibrant) DG Lie algebra, $\mathrm{HR}_*(\mathbb{L}_X, \mathfrak{g}) \;\cong\; \mathrm{H}_*[(\mathbb{L}_X)_{\mathfrak{g}}]$. The first assertion is then immediate from Theorem 6.2.2. The algebra isomorphism $(\mathbb{L}_X)_{\mathfrak{g}} \;\cong\; \mathrm{Sym}_k(\mathfrak{g}^* \otimes V)$ follows easily from formula (6.1.1).

The formula for the differential ∂ can follows easily from the fact that the universal representation $\rho : \mathbb{L}_X \to (\mathbb{L}_X)_{\mathfrak{g}} \otimes \mathfrak{g}$ is a *differential graded* Lie algebra homomorphism. $\square$

Example 1. Recall (see Example 5, [58, Ch. 24]) that the minimal Lie model for the complex projective space $\mathbb{CP}^r, r \geq 1$ is given by the free Lie algebra $\mathcal{L}_r := \mathbb{L}(v_1, v_2, \ldots, v_r)$ generated by $v_1, \ldots, v_r$, where the degree of v_i is $2i - 1$, and the differential is defined on generators by $dv_1 = 0$, $dv_i = \frac{1}{2} \sum_{j+k=i} [v_j, v_k]$ for all $i \geq 2$. By Corollary 6.2.5, we have

$$\mathrm{HR}_*(\mathbb{CP}^r, G) \cong \mathrm{H}_*[(\mathcal{L}_r)_{\mathfrak{g}}] \cong \mathrm{H}_*\big[\mathrm{Sym}\big(\bigoplus_{i=1}^{r} \mathfrak{g}^* \cdot v_i\big), \partial\big] ,$$

where $\mathfrak{g}^* \cdot v_i$ denotes a copy of $\mathfrak{g}^*$ in degree $2i - 1$ indexed by v_i and where the differential d is given on generators by

$$\partial(\xi^* \cdot v_i) = \sum_{j+k=i} (\xi_1^* \cdot v_j)(\xi_2^* \cdot v_k) .$$

Here, the cobracket on $\mathfrak{g}^*$ is given by $\xi^* \mapsto \xi_1^* \wedge \xi_2^*$ in Sweedler notation.

Example 2. As another application of Corollary 6.2.5, we can easily recover the result of Proposition 4.2.24. Indeed, let X be an n-connected space for some $n \geq 1$. Consider the minimal Quillen model $\mathbb{L}_X(V)$ of X. Then $V_i \cong \mathrm{H}_{i+1}(X; \mathbb{Q})$ for all $i \geq 0$. By the Rational Hurewicz Theorem, $\mathrm{H}_i(X, \mathbb{Q}) \cong \pi_i(X)_{\mathbb{Q}}$ for all $1 \leq i \leq 2n$. Hence, $V_i = 0$ for $i \leq n - 1$. Then the (nonzero) elements of $[V, V]$ must have homological degree $\geq 2n$, and therefore, by minimality of $\mathbb{L}_X$, $d(V_i) = 0$ for $n \leq i \leq 2n$. The differential ∂ on $(\mathbb{L}_X)_{\mathfrak{g}} = \mathrm{Sym}(\mathfrak{g}^* \otimes V)$ then vanishes on chains of degree $\leq 2n$, and the isomorphisms (4.2.25) are immediate from Corollary 6.2.5.

6.3 Proof of Comparison Theorem

Outline of the proof

The proof of Theorem 6.2.2 is based on several technical results. As observed in Section 6.3, $\mathrm{HR}_*(\mathcal{L}_X, \mathfrak{g}) \cong \mathrm{H}_*[N(L_{\mathfrak{g}})]$ for any semi-free simplicial Lie model L of X. Now, $L_{\mathfrak{g}} = \underline{R} \otimes_{\mathfrak{G}} \mathcal{O}(G)$, where $\underline{R}$ is the simplicial right $\mathfrak{G}$-module associated with the simplicial cocommutative Hopf algebra $R := UL$.

Our first step is to prove Theorem 6.3.1, which states that $\mathrm{HR}_*(X, G)$ is isomorphic to the homology of the derived tensor product $N(\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G)$. Recall that Quillen's rational homotopy theory gives a zig-zag of maps

$$\mathbb{Q}\mathbb{G}X \to \widehat{\mathbb{Q}}\mathbb{G}X \to \widehat{R} \leftarrow R\,,$$

where $\widehat{(-)}$ stands for completion with respect to the canonical augmentation. Here, the map g (which is by no means unique) is a weak equivalence in the model category of simplicial complete cocommutative Hopf algebras (sCHA's). The first and third arrows in the above zig-zag induce isomorphisms on all homotopy groups (see [136, Sec.3, Part I]). In Section 6.3, we use a relatively straightforward extension of the arguments of *loc. cit.* to verify (in a series of propositions) that the above zig-zag gives a zig-zag of the associated simplicial right $\mathfrak{G}$-modules where every arrow induces isomorphisms on all homotopy groups. The only subtlety here is that the notion of weak-equivalence in sCHA is *a priori* different from that of a map inducing an isomorphism on all homotopy groups (see [136, Sec.4, Part II]). This makes it necessary to argue that the map on simplicial right $\mathfrak{G}$-modules induced by g indeed induces isomorphisms on all homotopy groups. The proof of Theorem 6.3.1 follows easily from the verifications in Section 6.3 as well as Theorem 4.2.15.

Starting from Theorem 6.3.1, we proceed to argue in Section 6.3 that $\mathrm{HR}_*(X,G) \cong \mathrm{HR}_*(\mathcal{L}_X,\mathfrak{g})$ as graded vector spaces. The crucial ingredient is Proposition 6.3.6 (stated and proven in Section 6.3), which may well be of independent interest. This proposition states that if V is a vector space, then $\mathrm{Tor}_p^{\mathfrak{G}}(\underline{TV},\mathcal{O}(G)) = 0$ for $p > 0$, where $\underline{TV}$ is the right $\mathfrak{G}$-module corresponding to the cocommutative Hopf algebra $TV = ULV$, the universal enveloping algebra of the free Lie algebra generated by V. Surprisingly, the proof of Proposition 6.3.6 relies on topological arguments: it uses Theorem 6.3.1 as well as earlier computations of the representation homology of wedges of spheres (Proposition 5.3.2). We do not currently know a completely algebraic proof of this result. Proposition 6.3.6 also implies (see Proposition 6.3.10) another result that could be of independent interest: for any Lie algebra $\mathfrak{a}$, there is a natural isomorphism

$$\mathrm{HR}_*(\mathfrak{a},\mathfrak{g}) \cong \mathrm{Tor}_*^{\mathfrak{G}}(\underline{U\mathfrak{a}},\mathcal{O}(G))\,.$$

Finally, in Section 6.3, we show that the isomorphism $\mathrm{HR}_*(X,G) \cong \mathrm{HR}_*(\mathcal{L}_X,\mathfrak{g})$ of graded vector spaces constructed in Section 6.3 is indeed an isomorphism of graded commutative algebras. We do this by exhibiting for any $q \in \mathbb{N}$ a morphism of simplicial commutative algebras inducing the isomorphism $\mathrm{HR}_i(X,G) \cong \mathrm{H}_i[N(L_{\mathfrak{g}})]$ for $i \le q$. To show this, we first note that the canonical filtration (by powers of the augmentation ideal) on R induces a filtration on the right $\mathfrak{G}$-module $\underline{R}$. Then we use a generic connectivity argument due to Curtis [43, Sec. 4] to show that $\pi_q(F^r\underline{R}) = 0$ for $r > q$ (Proposition 6.3.11). This allows us to replace $\underline{R}$ with $\underline{R}/F^r\underline{R}$, $r \gg 0$ when computing homologies in degree $\le q$ of $N(\underline{R})\otimes^{\boldsymbol{L}}_{\mathfrak{G}}\mathcal{O}(G)$ (i.e, $\mathrm{HR}_i(X,G)$ for $i \le q$). Again as a consequence of Proposition 6.3.6, the n-simplices of $\underline{R}/F^r\underline{R}$ are right $\mathfrak{G}$-modules whose higher

Tor's with $\mathcal{O}(G)$ vanish. It follows from these facts that the composite map

$$\underline{\mathbb{Q}\mathbb{G}X} \otimes_{\mathfrak{G}} \mathcal{O}(G) \to \underline{\widehat{\mathbb{Q}}\mathbb{G}X} \otimes_{\mathfrak{G}} \mathcal{O}(G) \to \underline{\widehat{R}} \otimes_{\mathfrak{G}} \mathcal{O}(G) \to \underline{R}/F^r\underline{R} \otimes_{\mathfrak{G}} \mathcal{O}(G)$$

induces the isomorphism $\mathrm{HR}_i(X, G) \cong \mathrm{H}_i[L_{\mathfrak{g}}]$ for $i \leq q$ (on π_i's). It is not difficult to check that the maps above are morphisms of simplicial commutative algebras. This concludes our argument.

By Theorem 10.3.2 there are Quillen equivalences refining the Dold-Kan correspondence

$$N^* : \mathtt{DGLA}^+_{\mathbb{Q}} \rightleftarrows \mathtt{sLie}_{\mathbb{Q}} : N\,, \qquad N^* : \mathtt{DGA}^+_{\mathbb{Q}} \rightleftarrows \mathtt{sAlg}_{\mathbb{Q}} : N \qquad N^* : \mathtt{DGCA}^+_{\mathbb{Q}} \rightleftarrows \mathtt{scAlg}_{\mathbb{Q}} : N\,,$$

where $\mathtt{s}\mathscr{C}$ denotes the category of simplicial objects in a category $\mathscr{C}$. By Proposition 10.2.4, applying the functor N^* to a semi-free Quillen model of X gives a reduced semi-free simplicial Lie model of X. Let $L := L_X$ be a reduced semi-free simplicial Lie model of X. Consider the simplicial cocommutative Hopf algebra $R := U(L)$ as well as the simplicial complete cocommutative Hopf algebra $\widehat{R} \cong \widehat{U}(L)$ (where the completion is with respect to the canonical augmentation). These correspond to the right $\mathfrak{G}$-modules $\underline{R}$ and $\underline{\widehat{R}}$, which assign to $\langle m \rangle$ the the tensor product $R^{\otimes m}$ and the completed tensor product $\widehat{R}^{\widehat{\otimes} m}$ respectively.

Theorem 6.3.1. $\mathrm{HR}_*(X, G) \cong \mathrm{H}_*[N(\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G)]$.

The proof of Theorem 6.3.1 relies on several propositions from rational homotopy theory. These propositions, in turn, are based on the following lemma. Let V be a filtered reduced simplicial vector space. Let $\widehat{V}$ denote the completion of V with respect to the given filtration. More generally, for any $m \in \mathbb{N}$, one has the simplicial vector spaces $V^{\otimes m}, \widehat{V}^{\otimes m}$ and $V^{\widehat{\otimes} m} = \widehat{V}^{\widehat{\otimes} m}$, where $V^{\widehat{\otimes} m}$ denotes the completed tensor product $\varprojlim_r (V/F^rV)^{\otimes m}$. Let $\widehat{\mathrm{Sym}}^m(V)$ denote the image in $V^{\widehat{\otimes} m}$ of the symmetrization idempotent $e_m := \frac{1}{n!}\sum_{\sigma \in S_m} \sigma$. Let $\widehat{\mathrm{Sym}}(V) := \prod_{m=0}^{\infty} \widehat{\mathrm{Sym}}^m(V)$. Recall that a π_*-equivalence (see [115]) is a morphism inducing isomorphisms on all homotopy groups.

Lemma 6.3.2. *Suppose that for each $q > 0$, $\pi_q(F^rV) = 0$ for r sufficiently large. Then,*

(i) *For each $q > 0$, $\pi_q(F^r\widehat{V}) = 0$ for r sufficiently large.*

(ii) *The map $V^{\otimes m} \to \widehat{V}^{\widehat{\otimes} m}$ is a π_*-equivalences for all m.*

(iii) *The map $\mathrm{Sym}(V) \to \widehat{\mathrm{Sym}}(V)$ is a π_*-equivalence.*

Proof. By a long exact sequence of homotopy groups (LESH) argument, the natural map $\pi_q(V) \to \pi_q(V/F^rV)$ is an isomorphism for r sufficiently large. Thus, the inverse system $\{\pi_q(V/F^rV)\}$ is eventually constant. Thus, $\lim^1\{\pi_q(V/F^rV)\} = 0$. It follows from [136, Part I, Prop. 3.8] that $\pi_q(\widehat{V}) \cong \pi_q(V/F^rV)$ for r sufficiently large. Since $V/F^rV \cong \widehat{V}/F^k\widehat{V}$, we see that $\pi_q(\widehat{V}) \cong \pi_q(\widehat{V}/F^r\widehat{V})$ for r sufficiently large. Again by a LESH argument, $\pi_q(F^r\widehat{V}) = 0$ for r sufficiently large. This proves (i).

Moreover, by the Eilenberg-Zilber and Künneth Theorems, $\pi_q(V^{\otimes m}) \cong \pi_q[(V/F^rV)^{\otimes m}]$ for r sufficiently large (since the same is true for $m = 1$). It follows that the inverse system $\{\pi_q[(V/F^rV)^{\otimes m}]\}$ is eventually constant. Arguing as for the case when $m = 1$, we see that $\pi_q(\widehat{V}^{\widehat{\otimes m}}) \cong \pi_q[(V/F^rV)^{\otimes m}]$ for r sufficiently large. This proves that the natural map $V^{\otimes m} \to \widehat{V}^{\widehat{\otimes} m}$ induces an isomorphism on π_q for any fixed q. This proves (ii).

Since the map $V^{\otimes m} \to V^{\widehat{\otimes} m}$ is S_m-equivariant and since $\mathrm{Sym}^m(V)$ and $\widehat{\mathrm{Sym}}^m(V)$ are the images of the symmetrization idempotent e_m acting on

$V^{\otimes m}$ and $V^{\widehat{\otimes} m}$ respectively, the natural map $\mathrm{Sym}^m(V) \to \widehat{\mathrm{Sym}}^m(V)$ is a π_*-equivalence. Thus, the map $\mathrm{Sym}(V) \to \oplus_m \widehat{\mathrm{Sym}}^m(V)$ is a π_*-equivalence. Since V is reduced and by (ii), $\pi_q(V^{\widehat{\otimes} r}) = 0$ for $r > q$ (by the Eilenberg-Zilber and Künneth Theorems). It follows that $\pi_q(\oplus_{m \geq r} \widehat{\mathrm{Sym}}^m(V) = 0$ for $r > q$. Applying (ii) to $W := \oplus_m \widehat{\mathrm{Sym}}^m(V)$ with filtration given by $F^r W := \oplus_{m \geq r} \widehat{\mathrm{Sym}}^m(V)$, we see that the map $\oplus_m \widehat{\mathrm{Sym}}^m(V) \to \widehat{\mathrm{Sym}}(V)$ is a π_*-equivalence. This proves (iii). $\square$

Proposition 6.3.3. *The canonical map of $\mathfrak{G}$-modules $\underline{R} \to \underline{\widehat{R}}$ is a π_*-equivalence.*

Proof. It needs to be shown that the map $R^{\otimes m} \to \widehat{R}^{\widehat{\otimes} m}$ is a π_*-equivalence. By [136, Part I, Thm. 3.7], for any fixed q, $\pi_q(F^r R)$ vanishes for r sufficiently large. Lemma 6.3.2 (ii) then implies that the map $R^{\otimes m} \to \widehat{R}^{\widehat{\otimes} m}$ is a π_*-equivalence, as desired. $\square$

Recall that $\mathbb{G}X$ denotes the Kan loop group functor applied to a reduced simplicial/cellular model of X. Then, $\mathbb{Q}\mathbb{G}X$ is a simplicial cocommutative Hopf algebra equipped with a canonical augmentation. The completion $\widehat{\mathbb{Q}}\mathbb{G}X$ of $\mathbb{Q}\mathbb{G}X$ with respect to its canonical augmentation is a simplicial complete cocommutative Hopf algebra (sCHA). $\mathbb{Q}\mathbb{G}X$ as well as $\widehat{\mathbb{Q}}\mathbb{G}X$ correspond to simplicial right $\mathfrak{G}$-modules, which we denote by $\underline{\mathbb{Q}\mathbb{G}X}$ and $\underline{\widehat{\mathbb{Q}}\mathbb{G}X}$ respectively.

Proposition 6.3.4. *The map $\underline{\mathbb{Q}\mathbb{G}X} \to \underline{\widehat{\mathbb{Q}}\mathbb{G}X}$ is a π_*-equivalence.*

Proof. We need to show that for each m, the map $\mathbb{Q}\mathbb{G}X^{\otimes m} \to \widehat{\mathbb{Q}}\mathbb{G}X^{\widehat{\otimes} m}$ is a π_*-equivalence. By Lemma 6.3.2 (ii), this follows one we verify that for any fixed q, $\pi_q(F^r\mathbb{Q}\mathbb{G}X) = 0$ for r sufficiently large. This is immediate from [115, Thm. 4.72]. $\square$

We recall that the category $\mathtt{sCHA}$ of reduced sCHA's is a model category, whose cofibrant objects are retracts of semi-free sCHA's. The definition of semi-free sCHA is the obvious extension to the simplicial setting of the definition of a free complete cocommutative Hopf algebra: the free comple cocommutative Hopf algebra generated by a vector space V is $\widehat{TV}$, where V is primitive. We now apply Quillen's rational homotopy theory: in [136], Quillen proves several equivalences of homotopy categories (see *loc. cit*, pg. 211, Fig. 2) from which it follows that there is an isomorphism in $\mathrm{Ho}(\mathtt{sCHA})$ $\widehat{\mathbb{Q}}\mathbb{G}X \cong \widehat{R}$. By Theorem 4.7 of *loc. cit*, there is a morphism $g : \widehat{\mathbb{Q}}\mathbb{G}X \to \widehat{R}$ that is a simplicial homotopy equivalence. Denote the corresponding map of right $\mathfrak{G}$-modules by $\underline{g} : \underline{\widehat{\mathbb{Q}}\mathbb{G}X} \to \underline{\widehat{R}}$.

Proposition 6.3.5. *$\underline{g}$ is a π_*-equivalence.*

Proof. By [136, Part I, Theorem 3.7] and Lemma 6.3.2, the completion map $\widehat{R}^{\otimes m} \to \widehat{R}^{\widehat{\otimes} m}$ is a π_*-equivalence. Similarly, it can be shown that the map $\widehat{\mathbb{Q}}\mathbb{G}X^{\otimes m} \to \widehat{\mathbb{Q}}\mathbb{G}X^{\widehat{\otimes} m}$ is a π_* equivalence. To prove the desired lemma, we need to show that $\underline{g}^{\widehat{\otimes} m} : \widehat{\mathbb{Q}}\mathbb{G}X^{\widehat{\otimes} m} \to \widehat{R}^{\otimes m}$ is a π_* equivalence for each m. Since the diagram

$$\begin{tikzcd}
\widehat{\mathbb{Q}}\mathbb{G}X^{\otimes m} \arrow[r] \arrow[d, "g^{\otimes m}"'] & \widehat{\mathbb{Q}}\mathbb{G}X^{\widehat{\otimes} m} \arrow[d, "g^{\widehat{\otimes} m}"] \\
\widehat{R}^{\otimes m} \arrow[r] & \widehat{R}^{\widehat{\otimes} m}
\end{tikzcd}$$

commutes, it suffices (by the Eilenberg-Zilber and Künneth Theorems) to show that g is a π_*-equivalence. Let $\mathcal{P}$ denote the functor of primitive elements. By [136, Appendix A, Cor. 2.16], there is an isomorphism of simplicial vector spaces $\widehat{\mathrm{Sym}}(\mathcal{P}\widehat{R}) \xrightarrow{\sim} \widehat{R}$. For the same reason, $\widehat{\mathbb{Q}}\mathbb{G}X$ is isomorphic to $\widehat{\mathrm{Sym}}(\mathcal{P}\widehat{\mathbb{Q}}\mathbb{G}X)$ as simplicial vector spaces. Since $\mathcal{P}\widehat{R}$ is a canonical retract of $\widehat{R}$, $\pi_q(F^r\mathcal{P}\widehat{R}) = 0$ for r large enough (since the same holds for $\widehat{R}$) and for the same reason, $\pi_q(F^r\mathcal{P}\widehat{\mathbb{Q}}\mathbb{G}X) = 0$ for r sufficiently large. By Lemma 6.3.2 (iii), the horizontal arrows in the commutative diagram below are π_*-equivalences.

$$\begin{tikzcd}
\mathrm{Sym}(\mathcal{P}\widehat{\mathbb{Q}}\mathbb{G}X) \arrow[r] \arrow[d, "\mathrm{Sym}(\mathcal{P}g)"'] & \widehat{\mathrm{Sym}}(\mathcal{P}\widehat{\mathbb{Q}}\mathbb{G}X) \cong \widehat{\mathbb{Q}}\mathbb{G}X \arrow[d, "g"] \\
\mathrm{Sym}(\mathcal{P}\widehat{R}) \arrow[r] & \widehat{\mathrm{Sym}}(\mathcal{P}\widehat{R}) \cong \widehat{R}
\end{tikzcd}$$

By [136, Part II, Theorem 4.7], $\mathcal{P}g$ is a π_*-equivalence. Thus, the left vertical arrow in the above diagram is a π_*-equivalence. It follows that g is a π_*-equivalence, as desired. □

Proof. (Proof of Theorem 6.3.1) By Proposition 6.3.3, Proposition 6.3.4 and Proposition 6.3.5, there is a diagram of complexes of right $\mathfrak{G}$-modules in which each arrow is a quasi-isomorphism

$$N(\underline{\mathbb{Q}\mathbb{G}X}) \longrightarrow N(\underline{\widehat{\mathbb{Q}}\mathbb{G}X}) \xrightarrow{N(g)} N(\underline{\widehat{R}}) \longleftarrow N(\underline{R}) \ .$$

Hence, there is an isomorphism in the derived category $\mathcal{D}(\mathbb{Q})$ of complexes of $\mathbb{Q}$-vector spaces

$$N(\underline{\mathbb{Q}\mathbb{G}X}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \cong N(\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \,.$$

The desired result now follows from Theorem 4.2.15. □

Let V be a $\mathbb{Q}$-vector space and let $\underline{TV}$ denote the right $\mathfrak{G}$-module corresponding to the cocommutative Hopf algebra $TV \cong U(LV)$. The module $\underline{TV}$ has a weight grading induced by the weight grading on TV in which V has weight 1. Let $(\underline{TV})_q$ denote the component of $\underline{TV}$ of weight q. For example, $\underline{V} := \underline{TV}_1$ is the $\mathfrak{G}$-module defined by $\mathtt{lin}^*_{\mathbb{Q}} \otimes V$, see (4.2.26). The following proposition may be of independent interest.

Proposition 6.3.6. *Let G be an affine algebraic group over $\mathbb{Q}$, with Lie algebra $\mathfrak{g}$. Then*

$$\mathrm{Tor}_i^{\mathfrak{G}}(\underline{TV},\, \mathcal{O}(G)) \cong \begin{cases} \mathrm{Sym}(\mathfrak{g}^* \otimes V) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

In particular, $\mathrm{Tor}_i^{\mathfrak{G}}[(\underline{TV})_q, \mathcal{O}(G)] = 0$ *for all $i > 0$ and $q \geq 0$.*

The following lemma (see [100, Prop. 4.3]) is essential for the proof of Proposition 6.3.6. For the benefit of the reader, we outline a proof of this lemma that is different from that given in *loc. cit.*.

Lemma 6.3.7. *For any $\mathfrak{a} \in \mathtt{Lie\,Alg}_k$, there is a natural isomorphism of commutative algebras*

$$U\mathfrak{a} \otimes_{\mathfrak{G}} \mathcal{O}(G) \;\cong\; \mathfrak{a}_{\mathfrak{g}}\,,$$

where $\mathfrak{a}_{\mathfrak{g}}$ is the representation algebra defined in (6.1.1).

Proof. Let $B \in \mathtt{CommAlg}_k$. From the left $\mathfrak{G}$-module $\mathcal{O}(G)$, one can form the right $\mathfrak{G}$-module $\mathrm{Hom}_k(\mathcal{O}(G), B)$, which assigns $\mathrm{Hom}_k(\mathcal{O}(G)^{\otimes m}, B)$ to $\langle m \rangle$. Since B is a commutative k-algebra and since $\mathcal{O}(G)$ is a strictly monoidal left $\mathfrak{G}$-module, $\mathrm{Hom}_k(\underline{\mathcal{O}(G)}, B)$ acquires the structure of a lax monoidal right $\mathfrak{G}$-module. This structure is given by the maps

$$\mathrm{Hom}_k(\mathcal{O}(G)^{\otimes m}, B) \otimes \mathrm{Hom}_k(\mathcal{O}(G)^{\otimes n}, B) \xrightarrow{\mu_B \circ (-\otimes -)} \mathrm{Hom}_k(\mathcal{O}(G)^{\otimes (m+n)}, B)\,,$$

where μ_B is the product on B. By the standard $\mathrm{Hom} - \otimes$ adjunction, there is a natural isomorphism of k-vector spaces

$$\mathrm{Hom}_k(U\mathfrak{a} \otimes_{\mathfrak{G}} \mathcal{O}(G), B) \;\cong\; \mathrm{Hom}_{\mathtt{Mod}\text{-}\mathfrak{G}}(U\mathfrak{a}, \mathrm{Hom}_k(\mathcal{O}(G), B))\,.$$

It is a routine verification to check that under this isomorphism, the commutative k-algebra homomorphisms from $U(LV) \otimes_{\mathfrak{G}} \mathcal{O}(G)$ to B correspond to the right $\mathfrak{G}$-module homomorphisms from $U\mathfrak{a}$ to $\mathrm{Hom}_k(\mathcal{O}(G), B)$ that respect the (lax) monoidal structure. Since $\mathcal{O}(G)$ is a coalgebra and B is an algebra, $\mathrm{Hom}_k(\mathcal{O}(G), B)$ has an algebra structure (with product given by convolution). Another routine verification shows that the set of right $\mathfrak{G}$-module homomorphisms from $U\mathfrak{a}$ to $\mathrm{Hom}_k(\mathcal{O}(G), B)$ that respect the (lax) monoidal structure is in (natural) bijection with the set of k-algebra homomorphisms φ from $U\mathfrak{a}$ to $\mathrm{Hom}_k(\mathcal{O}(G), B)$ that satisfy the following additional conditions:

$$\varphi(x)(fg) \,=\, \varphi(x^{(1)})(f)\varphi(x^{(2)})(g)\,,\;\; \varphi(x)(1_{\mathcal{O}(G)}) \,=\, \varepsilon(x)1_B\,,\;\; \varphi(Sx)(f) \,=\, \varphi(x)(Sf)$$

for all $x \in U\mathfrak{a}$ and $f, g \in \mathcal{O}(G)$. Here, ε and S stand for the counit and antipode of $U\mathfrak{a}$ respectively the coproduct in $U\mathfrak{a}$ is given by $x \mapsto x^{(1)} \otimes x^{(2)}$ in Sweedler notation. It is not difficult to verify that the third condition above follows from the first two. As shown in [116, Example 3.4], the algebra homomorphisms from $U\mathfrak{a}$ to $\mathrm{Hom}_k(\mathcal{O}(G), B)$ satisfying the above conditions are in natural bijection with Lie algebra homomorphisms from $\mathfrak{a}$ to $\mathfrak{g}(B)$. Indeed, φ satisfies these conditions for all x in $U\mathfrak{a}$ iff it satisfies these conditions for $x \in \mathfrak{a}$. For $x \in \mathfrak{a}$, these conditions are equivalent to the assertion that $\varphi(x)$ is a k-linear derivation on $\mathcal{O}(G)$ with respect to the homomorphism $1_B \circ \varepsilon_{\mathcal{O}(G)}$, where $\varepsilon_{\mathcal{O}(G)}$ denotes the canonical augmentation on $\mathcal{O}(G)$. Such derivations are indeed in bijection with elements of $\mathrm{Hom}_k(\mathfrak{g}^*, B) \cong \mathfrak{g}(B)$. We thus, have a natural bijection

$$\mathrm{Hom}_{\mathtt{CommAlg}_k}(U\mathfrak{a} \otimes_{\mathfrak{G}} \mathcal{O}(G), B) \,\cong\, \mathrm{Hom}_{\mathtt{Lie\,Alg}_k}(\mathfrak{a}, \mathfrak{g}(B))\,.$$

The desired lemma now follows from the Yoneda lemma. □

Proof of Proposition 6.3.6. Since $\underline{TV} \otimes_{\mathfrak{G}} \mathcal{O}(G) \,\cong\, U(LV) \otimes_{\mathfrak{G}} \mathcal{O}(G)$, the required isomorphism for $i = 0$ follows from Lemma 6.3.7. To prove the vanishing of $\mathrm{Tor}_i^{\mathfrak{G}}(\underline{TV},\, \mathcal{O}(G))$ for $i > 0$, we assign V (homological) degree 2. Then $\underline{TV}$ is a graded right $\mathfrak{G}$-module, whose component in degree $2q$ is $(\underline{TV})_q$. Thus,

$$\mathrm{H}_n[\underline{TV} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G)] \;\cong\; \bigoplus_{2q+i=n} \mathrm{Tor}^i_{\mathfrak{G}}[(\underline{TV})_q, \mathcal{O}(G)]\,.$$

The desired proposition will follow once we show that

$$\mathrm{H}_*[\underline{TV} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G)] \;\cong\; \mathrm{Sym}(\mathfrak{g}^* \otimes V)\,. \tag{6.3.8}$$

Equip $N^*TV \cong T(N^{-1}V)$ (see formula 10.2.3 in Section 10) with the simplicial cocommutative Hopf algebra structure given by its identification with $UL(N^{-1}V)$. This gives N^*TV the structure of a simplicial right $\mathfrak{G}$-module (which we denote by $\underline{N^*TV}$). This module assigns to the free group $\langle m\rangle$ the simplicial vector space $N^*TV^{\otimes m}$. Note that since V has degree 2, $N^*LV \cong L(N^{-1}V)$ is a semi-free simplicial Lie model for the space X given by a wedge of $(\dim_k V)$ 3-spheres. By Theorem 6.3.1 and Proposition 5.3.2,

$$\mathrm{H}_*[N(\underline{N^*TV}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G)] \;\cong\; \mathrm{HR}_*(X, G) \;\cong\; \mathrm{Sym}(\mathfrak{g}^* \otimes V)\,.$$

The desired proposition is therefore, a consequence of Lemma 6.3.9 below.

$\square$

Lemma 6.3.9. *The unit of the adjunction $\varepsilon : TV \to N(N^*TV)$ induces a quasi-isomorphism of right $\mathfrak{G}$-modules $\underline{\varepsilon} : \underline{TV} \to N(\underline{N^*TV})$.*

Proof. The morphism $\underline{\varepsilon} : \underline{TV} \to N(\underline{N^*TV})$ is defined by the family of maps

$$\underline{\varepsilon}(\langle m\rangle) :\; TV^{\otimes m} \xrightarrow{\epsilon^{\otimes m}} N(N^*TV)^{\otimes m} \to N(N^*TV^{\otimes m})\,,$$

where the last arrow is the Eilenberg-Zilber map (which is well-defined for $m > 2$ because of the associativity of the Eilenberg-Zilber map for $m = 2$). That this is a quasi-isomorphism follows from the Künneth Theorem and the fact that $\varepsilon : TV \to N(N^*TV)$ is a quasi-isomorphism. It follows from the associativity of the Eilenberg-Zilber map that the maps $\underline{\varepsilon}(\langle m\rangle)$ indeed assemble into a morphism of right $\mathfrak{G}$-modules. $\square$

We now show that $\mathrm{HR}_*(X,G) \cong \mathrm{HR}_*(\mathcal{L}_X,\mathfrak{g})$ as graded vector spaces. Without loss of generality, we may assume that $\mathcal{L}_X$ is a semi-free DG Lie model of X. Since the representation functor $(-)_\mathfrak{g}$ is left adjoint, it commutes with N^*, i.e, there is a commutative diagram of functors

$$\begin{tikzcd} \mathtt{DGLA}_{\mathbb{Q}} \arrow[r, "N^*"] \arrow[d, "(-)_\mathfrak{g}"'] & \mathtt{sLie}_{\mathbb{Q}} \arrow[d, "(-)_\mathfrak{g}"] & \\ \mathtt{DGCA}_{\mathbb{Q}} & {} \arrow[r, "N^*"] & \mathtt{scAlg}_{\mathbb{Q}} \end{tikzcd}$$

Thus, $\mathrm{HR}_*(\mathcal{L}_X,\mathfrak{g}) \cong \mathrm{H}_*[N(L_\mathfrak{g})]$, where $L := N^*\mathcal{L}_X$. By Lemma 6.3.7, $L_\mathfrak{g} \cong \underline{R} \otimes_{\mathfrak{G}} \mathcal{O}(G)$, where $R := UL$. Thus, $\mathrm{HR}_*(\mathcal{L}_X,\mathfrak{g}) \cong \mathrm{H}_*[\mathrm{C}(\underline{R} \otimes_{\mathfrak{G}} \mathcal{O}(G))] \cong \mathrm{H}_*[\mathrm{C}(\underline{R}) \otimes_{\mathfrak{G}} \mathcal{O}(G)]$, where C stands for associated chain complex. Since, L is a semi-free simplicial Lie model of X, the right $\mathfrak{G}$-module of n-simplices in the simplicial right $\mathfrak{G}$-module $\underline{R}$ is of the form $\underline{TV}$ for some vector space V. It follows from Proposition 6.3.6 that $\mathrm{C}(\underline{R})$ is a complex of right $\mathfrak{G}$-modules whose higher Tor's with $\mathcal{O}(G)$ vanish. Thus, the map $\mathrm{C}(\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \to \mathrm{C}(\underline{R}) \otimes_{\mathfrak{G}} \mathcal{O}(G)$ is a quasi-isomorphism. Note that there is a quasi-isomorphism of complexes of right $\mathfrak{G}$-modules $N(\underline{R}) \to \mathrm{C}(\underline{R})$. That $\mathrm{HR}_*(X,G) \cong \mathrm{HR}_*(\mathcal{L}_X,\mathfrak{g})$ as graded vector spaces follows from Theorem 6.3.1, by which $\mathrm{HR}_*(X,G) \cong \mathrm{H}_*[N(\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G)]$.

As yet another consequence of Proposition 6.3.6, we have the following result which may be of independent interest.

Proposition 6.3.10. *For any Lie algebra $\mathfrak{a} \in \mathtt{LieAlg}_{\mathbb{Q}}$, there is a natural isomorphism*

$$\mathrm{HR}_*(\mathfrak{a}, \mathfrak{g}) \cong \mathrm{Tor}_*^{\mathfrak{G}}(\underline{U\mathfrak{a}},\, \mathcal{O}(G)) \,.$$

Proof. As argued above, it suffices to show that $\mathrm{HR}_*(\mathfrak{a}, \mathfrak{g}) \cong \pi_*[\mathcal{L}_{\mathfrak{g}}]$ for any semi-free simplicial resolution $\mathcal{L}$ of $\mathfrak{a}$. By Lemma 6.3.7, $\mathcal{L}_{\mathfrak{g}} \cong \underline{U\mathcal{L}} \otimes_{\mathfrak{G}} \mathcal{O}(G)$. Since $\mathcal{L}$ is semi-free, the right $\mathfrak{G}$-module of n-simplices in the simplicial right $\mathfrak{G}$-module $\underline{U\mathcal{L}}$ is of the form $\underline{TV}$ for some vector space V. It follows from Proposition 6.3.6 that the map $\mathrm{C}(\underline{U\mathcal{L}}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \to \mathrm{C}(\underline{U\mathcal{L}}) \otimes_{\mathfrak{G}} \mathcal{O}(G)$ is a quasi-isomorphism. The desired proposition then follows once we establish that $\underline{U\mathcal{L}}$ is a simplicial resolution of $\underline{U\mathfrak{a}}$. For this, we need to check that for any m, $U\mathcal{L}^{\otimes m}$ resolves $U\mathfrak{a}^{\otimes m}$. This follows from the Eilenberg-Zilber and Künneth Theorems. $\square$

To complete the proof of Theorem 6.2.2, it remains to show that $\mathrm{HR}_*(X, G) \cong \mathrm{HR}_*(\mathcal{L}_X, \mathfrak{g})$ as graded $\mathbb{Q}$-algebras. For this, given any $r \in \mathbb{N}$, we shall produce a morphism of simplicial commutative algebras that induces the isomorphism $\mathrm{HR}_q(X, G) \cong \mathrm{H}_q[N(L_{\mathfrak{g}})]$ for $q < r$.

Recall that $R := UL$ is a semi-free simplicial associative algebra filtered by powers of its augmentation ideal. This filtration induces a filtration on the simplicial right $\mathfrak{G}$-module $\underline{R}$: if the algebra of n-simplices of R is TV for some vector space V, then the right $\mathfrak{G}$-module of n-simplices of $F^r\underline{R}$ is $\oplus_{q \geq r}(\underline{TV})_q$. The following connectivity result holds for the filtered right $\mathfrak{G}$-module $\underline{R}$.

Proposition 6.3.11. *For $r > q$, we have $\pi_q(F^r\underline{R}) = 0$.*

Proof. It needs to be shown that for all $\langle m \rangle$, $\pi_q(F^r\underline{R}(\langle m \rangle)) = 0$ for $r > q$. For

$m = 0$, this is obvious. For $m = 1$, this is [136, Part I, Thm. 3.7]. For arbitrary m, we generalize the argument in *loc. cit.*. The functor $\mathtt{Lie}_{\mathbb{Q}} \to \mathtt{Vect}_{\mathbb{Q}}$, $L \mapsto F^r \underline{UL}(\langle m \rangle)$ takes 0 to 0 and commutes with direct limits. By [43, Remark 4.10], the arguments in Section 4 of *loc. cit.* proving Lemma (2.5) therein apply to this functor as well. It therefore, suffices to verify the desired proposition for $R = U\mathfrak{l}$, where $\mathfrak{l}$ is the free simplicial Lie algebra generated by $V := \overline{\mathbb{Q}}K$, where K is a finite wedge sum of simplicial circles. Note that in this case, $R = TV$, and V is a *connected* simplicial vector space. In this case, $F^r \underline{R}(\langle m \rangle) = \oplus_{r_1 + \ldots + r_m \geq r} V^{\otimes r_1} \otimes \ldots \otimes V^{\otimes r_m}$. That π_q of each summand vanishes for $q < r$ follows from the Eilenberg-Zilber and Künneth Theorems. This proves the desired proposition. $\square$

Proposition 6.3.12. *For r sufficiently large, all arrows in the following commutative diagram induce isomorphisms on the homology groups $\mathrm{H}_i[-]$, $i \leq q$.*

$$\begin{tikzcd}
\mathrm{C}(\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \arrow[r] \arrow[d] & \mathrm{C}(\underline{R}/F^r\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \arrow[d] \\
\mathrm{C}(\underline{R}) \otimes_{\mathfrak{G}} \mathcal{O}(G) \arrow[r] & \mathrm{C}(\underline{R}/F^r\underline{R}) \otimes_{\mathfrak{G}} \mathcal{O}(G)
\end{tikzcd}$$

Proof. Both $\mathrm{C}(\underline{R})$ and $\mathrm{C}(\underline{R}/F^r\underline{R})$ are complexes of right $\mathfrak{G}$-modules whose higher Tors with $\mathcal{O}(G)$ vanish by Proposition 6.3.6. It follows that the vertical arrows induce isomorphisms on all homology groups. It therefore, suffices to show that the horizontal arrow on top of the above diagram induces isomorphisms on $\mathrm{H}_i[-]$, $i \leq q$ for r sufficiently large.

Consider the good truncation $\tau_{\geq q+1}\mathrm{C}$ (see [164, Sec. 1.2.7]) of a chain complex C of right $\mathfrak{G}$-modules. The exact sequence $0 \to \tau_{\geq q+1}\mathrm{C} \to \mathrm{C} \to \tau_{< q+1}\mathrm{C} \to 0$ of complexes of right $\mathfrak{G}$-modules gives a distinguished triangle in $\mathcal{D}(\mathbb{Q})$ for any right $\mathfrak{G}$-module N.

$$\tau_{\geq q+1}\mathrm{C} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} N \to \mathrm{C} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} N \to \tau_{<q+1}\mathrm{C} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} N \to \tau_{\geq q+1}\mathrm{C} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} N[1]$$

It is easy to see that $\mathrm{H}_i(\tau_{\geq q+1}\mathrm{C} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} N) = 0$ for $i < q+1$. The long exact sequence of homologies associated with the above distinguished triangle then implies that

$$\mathrm{H}_i(\mathrm{C} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} N) \cong \mathrm{H}_i(\tau_{<q+1}\mathrm{C} \otimes^{\boldsymbol{L}}_{\mathfrak{G}} N) \text{ for } i \leq q\,. \tag{6.3.13}$$

By Proposition 6.3.11, the map $\tau_{<q+1}\mathrm{C}(\underline{R}) \to \tau_{<q+1}\mathrm{C}(\underline{R}/F^r\underline{R})$ is a quasi-isomorphism for $r > q$. Thus, the map $\mathrm{H}_*[\tau_{<q+1}\mathrm{C}(\underline{R}) \otimes^{L}_{\mathfrak{G}} \mathcal{O}(G)] \to \mathrm{H}_*[\tau_{<q+1}\mathrm{C}(\underline{R}/F^r\underline{R}) \otimes^{L}_{\mathfrak{G}} \mathcal{O}(G)]$ is an isomorphism of graded $\mathbb{Q}$-vector spaces. The desired proposition now follows from (6.3.13). $\square$

Note that the filtration on $\widehat{R}$ induces a filtration on the right $\mathfrak{G}$-module $\underline{\widehat{R}}$. Clearly, $\underline{R}/F^r\underline{R} \cong \underline{\widehat{R}}/F^r\underline{\widehat{R}}$. The following corollary is immediate from Proposition 6.3.3 and Proposition 6.3.12.

Corollary 6.3.14. *For r sufficiently large, all arrows in the following commutative diagram induce isomorphisms on the homology groups $\mathrm{H}_i[-]$, $i \leq q$.*

$$\begin{tikzcd}
\mathrm{C}(\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \arrow[r] \arrow[d] & \mathrm{C}(\underline{\widehat{R}}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \arrow[dl] \\
\mathrm{C}(\underline{R}/F^r\underline{R}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) &
\end{tikzcd}$$

Recall that there is a weak equivalence between cofibrant objects in $\mathtt{sCHA}$ $g : \widehat{\mathbb{Q}}\mathbb{G}X \to \widehat{R}$ inducing a map of simplicial right $\mathfrak{G}$-modules $\underline{g}$ (see Proposition 6.3.5). Consider the following commutative diagram, where the second arrow on the top and bottom rows is induced by $\underline{g}$.

$$
\begin{tikzcd}
\mathrm{C}(\underline{\mathbb{Q}\mathbb{G}X}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \arrow[r] \arrow[d] & \mathrm{C}(\underline{\widehat{\mathbb{Q}}\mathbb{G}X}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) \arrow[r] \arrow[d] \arrow[dr] & \mathrm{C}(\widehat{\underline{R}}) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G) & \\
\mathrm{C}[\underline{\mathbb{Q}\mathbb{G}X} \otimes_{\mathfrak{G}} \mathcal{O}(G)] \arrow[r] & \mathrm{C}[\underline{\widehat{\mathbb{Q}}\mathbb{G}X} \otimes_{\mathfrak{G}} \mathcal{O}(G)] \arrow[r] & \mathrm{C}[\widehat{\underline{R}} \otimes_{\mathfrak{G}} \mathcal{O}(G)] \arrow[r] & \mathrm{C}[\underline{R}/F^r\,\underline{R} \otimes_{\mathfrak{G}} \mathcal{O}(G)]
\end{tikzcd}
\tag{6.3.15}
$$

By Proposition 4.2.15 the left vertical arrow in (6.3.15) induces isomorphisms on all homologies. The two arrows on the top row of (6.3.15) induce isomorphisms on all homologies by Propositions 6.3.4 and 6.3.5 respectively. The diagonal arrow induces isomorphisms on $\mathrm{H}_i[-]$, $i \leq q$ for r sufficiently large by Proposition 6.3.12 and Corollary 6.3.14. An isomorphism $\mathrm{HR}_i(X, G) \cong \mathrm{H}_i[N(L_{\mathfrak{g}})]$, $i \leq q$ is thus induced on homologies (for sufficiently large r) by the composition of the maps on the bottom row of (6.3.15). That the composition of maps in the bottom row is a map of DG commutative algebras follows from the fact that each of the maps

$$
\underline{\mathbb{Q}\mathbb{G}X} \otimes_{\mathfrak{G}} \mathcal{O}(G) \;\to\; \underline{\widehat{\mathbb{Q}}\mathbb{G}X} \otimes_{\mathfrak{G}} \mathcal{O}(G) \;\to\; \widehat{\underline{R}} \otimes_{\mathfrak{G}} \mathcal{O}(G) \;\to\; \underline{R}/F^r\,\underline{R} \otimes_{\mathfrak{G}} \mathcal{O}(G)
$$

is a morphism of simplicial commutative algebras. Indeed, this last fact follows from [100, Prop. 3.4] and the facts that $\mathcal{O}(G)$ is a lax-monoidal left $\mathfrak{G}$-module, the n-simplices of the right $\mathfrak{G}$-modules $\underline{\mathbb{Q}\mathbb{G}X}$, $\underline{\widehat{\mathbb{Q}}\mathbb{G}X}$, $\widehat{\underline{R}}$ and $\underline{R}/F^r\,\underline{R}$ are lax-monoidal for each n, and the morphisms

$$
\underline{\mathbb{Q}\mathbb{G}X} \;\to\; \underline{\widehat{\mathbb{Q}}\mathbb{G}X} \;\to\; \widehat{\underline{R}} \;\to\; \underline{R}/F^r\,\underline{R}
$$

are natural transformations of lax-monoidal functors on n-simplices for each n. This completes the proof of Theorem 6.2.2.

Remark.

The results of this section go through with $\mathbb{Q}$ replaced by any field k of characteristic 0. Indeed, the proofs of Propositions 6.3.3 and 6.3.4 work for any such field k. For Proposition 6.3.5, we work with a semi-free simplicial Lie model L of X over $\mathbb{Q}$. The corresponding Lie model over k is $L \otimes_{\mathbb{Q}} k$. The corresponding sCHA over k is $\widehat{R \otimes_{\mathbb{Q}} k}$. The π_*-equivalence of sCHA's (over $\mathbb{Q}$) $f \,:\, \widehat{R} \to \widehat{\mathbb{Q}}\mathbb{G}X$ extends to a π_*-equivalence of sCHA's over k $f \,:\, \widehat{R \otimes_{\mathbb{Q}} k} \to \widehat{k}\mathbb{G}X$. This proves Proposition 6.3.5 over k. Theorem 6.3.1, Proposition 6.3.6, Theorem 6.2.2 and Proposition 6.3.10 can then be proven over k as done above (over $\mathbb{Q}$).

6.4 Complex projective spaces

In this section, we assume that $k = \mathbb{C}$, and G is a complex *reductive* group of finite rank l. We denote by $\mathfrak{g}$ the Lie algebra of G. Recall that for any homogeneous invariant polynomial $P \in I^d(\mathfrak{g})$, one has the associated Drinfeld trace map (see formula (6.1.12))

$$\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L}) :\ \mathrm{HC}_*^{(d)}(\mathcal{L}) \ \to\ \mathrm{HR}_*(\mathcal{L}, \mathfrak{g})^G \,.$$

The Drinfeld trace maps associated with a set $\{P_1, \ldots, P_l\}$ of homogeneous generators of $I(\mathfrak{g})$ assemble to a homomorphism of graded commutative algebras

$$\mathrm{Sym}\,\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L}) :\ \mathrm{Sym}_k\Big(\bigoplus_{i=1}^{l} \mathrm{HC}_*^{(d_i)}(\mathcal{L})\Big) \ \to\ \mathrm{HR}_*(\mathcal{L}, \mathfrak{g})^G \,, \tag{6.4.1}$$

where $d_i = \deg(P_i)$ are the fundamental degrees of the Lie algebra $\mathfrak{g}$.

If $\mathcal{L} = \mathcal{L}_X$ is a Quillen Lie model of a simply connected space X, we refer to the map (6.4.1) as a *Drinfeld homomorphism* for X. With this terminology, we can now state the main result of this section.

Theorem 6.4.2. *For $X = \mathbb{CP}^r$ $(r \geq 1)$, the Drinfeld homomorphism* (6.4.1) *is an isomorphism.*

Proof. As shown in Example 1, there is an isomorphism of graded commutative algebras
$\mathrm{HR}_*(\mathbb{CP}^r, G) \cong \mathrm{H}_*[(\mathcal{L}_r)_{\mathfrak{g}}]$, where $\mathcal{L}_r$ is the minimal Lie model of $\mathbb{CP}^r$. Further, all (quasi)-isomorphisms in the proof of Theorem 6.2.2 are G-equivariant. Indeed, the G-action on all complexes involved in the proof of Theorem 6.2.2 is induced by the G-action on the left $\mathfrak{G}$-module $\mathcal{O}(G)$ coming from the conjugation action of G on itself. Since G is reductive, it follows that $\mathrm{HR}_*(\mathbb{CP}^r, G)^G \cong \mathrm{H}_*[(\mathcal{L}_r)_{\mathfrak{g}}^G]$ (as graded vector spaces). On the other hand, $\mathcal{L}_r$ is Koszul dual to the graded linear dual of the *cohomologically graded* commutative algebra $\mathbb{C}[u]/(u^{r+1})$, where u has cohomological degree 2. By [24, Thm. 6.5 (b)], $\mathrm{H}_*[(\mathcal{L}_r)_{\mathfrak{g}}^G] \cong \mathrm{H}^{-*}[\mathfrak{g}[u]/(u^{r+1}), \mathfrak{g}; \mathbb{C}]$.

A related cohomology is computed in [60, Thm. A.], which states that if z has cohomological degree 0, then $\mathrm{H}^{-*}[\mathfrak{g}[z]/(z^{r+1}), \mathfrak{g}; \mathbb{C}]$ is the free exterior algebra on $r \cdot l$ generators of cohomological degree $2m+1$ and z-weights $-m(r+1)-1, -m(r+1)-2, \ldots, -m(r+1)-r$, where m runs over the exponents $m = 1, \ldots, m_l$ of $\mathfrak{g}$. Here. the (relative) Lie algebra cohomology being considered in 'continuous' Lie algebra cohomology. In fact, setting $\mathfrak{a}$ to be the semi-free DG Lie algebra Koszul dual to $\mathbb{C}[z]/(z^{r+1})$, the results of *loc. cit.* imply that the Drinfeld trace $\mathrm{Sym}(\mathrm{Tr}_{\mathfrak{g}})$ is an isomorphism for $\mathfrak{a}$. In fact, for $d = d_1, d_2, \ldots, d_l$, $\mathrm{HC}_*^{(d)}(\mathfrak{a})$ is a vector space in homological degree $-2m-1$ (where $m = d-1$) having a basis of r elements with z-weights $-m(r+1)-1, -m(r+1)-2, \ldots, -m(r+1)-r$.

The claim then follows from the observation that the Drinfeld trace for $\mathcal{L}_r$

and the Drinfeld trace for $\mathfrak{a}$ are identical as morphisms of $\mathbb{Z}_2$-graded commutative algebras, though they differ as morphisms of $\mathbb{Z}$-graded commutative algebras. This argument is identical to that deducing [24, Thm 9.1] from the corresponding result in [60]. $\square$

Remark 6.4.3. It follows from [25, Thm. 3.2 and Prop. A.3] that under the identification $\mathrm{H}_*[(\mathcal{L}_r)_{\mathfrak{g}}^G] \cong \mathrm{H}^{-*}[\mathfrak{g}[u]/(u^{r+1}), \mathfrak{g}; \mathbb{C}]$ mentioned in the above proof, the Drinfeld trace map is identified with the map [158, Equation (3.1)] given by Teleman with an explicit formula (see *loc. cit.*, Equation (2.2)). The latter map is implicitly described in the work of Feigin [62].

As a consequence of Theorem 6.4.2, we get the following explicit description of (the invariant part of) representation homology of complex projective spaces.

Corollary 6.4.4. *Let G be a complex reductive Lie group of rank l. Let $m_1, \ldots, m_l$ denote the exponents of the Lie algebra of G. Then, for $r \geq 1$, there is an isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(\mathbb{CP}^r, G)^G \cong \mathrm{Sym}(\xi_1^{(i)}, \xi_3^{(i)}, \ldots, \xi_{2r-1}^{(i)} \, : \, i = 1, 2, \ldots, l) \,,$$

where each generator $\xi_{2s-1}^{(i)}$ has homological degree $2rm_i + 2s - 1$. In the limit, we have

$$\mathrm{HR}_*(\mathbb{CP}^\infty, G)^G \cong \mathbb{C} \,.$$

Proof. Note that, under the isomorphism of $\mathbb{Z}_2$-graded vector spaces $\mathrm{HC}_*^{(d)}(\mathfrak{a}) \cong \mathrm{HC}_*^{(d)}(\mathcal{L}_r)$, a basis element of homological degree $-2m-1$ and z-weight $-m(r+1) - i$ corresponds to a basis element of homological degree $2[m(r+1)+i] - 2m - 1$ and u-weight $-m(r+1) - i$ (since u has homological degree -2. This shows that $\mathrm{HC}_*^{(d)}(\mathcal{L}_r)$ has r basis elements of homological degree $2mr+1$, $2mr+3$, $\ldots$, $2mr+2r-1$ for each exponent m of $\mathfrak{g}$. Hence, the first statement follows from Theorem 6.4.2. The second statement follows from the first, if we notice that the map $\mathrm{HR}_*(\mathbb{CP}^r, G)^G \to \mathrm{HR}_*(\mathbb{CP}^s, G)^G$ induced by any map of spaces $\mathbb{CP}^r \to \mathbb{CP}^s$ vanishes for $s \gg r$ for degree reasons. $\square$

6.5 Relation to $\mathbb{S}^1$-equivariant homology

If $\mathcal{L} = \mathcal{L}_X$ is a Lie model of a simply connected space X of finite rational type, the Lie-Hodge decomposition (6.1.7) has a natural topological interpretation described in [26]. This interpretation is based on a well-known theorem of Jones [86] that identifies the (reduced) cyclic homology $\mathrm{HC}_*(U\mathcal{L}_X)$ of the universal enveloping algebra of $\mathcal{L}_X$ with the (reduced) $\mathbb{S}^1$-equivariant homology of the free loop space $\mathscr{L}X$ of X

$$\mathrm{HC}_*(U\mathcal{L}_X) \cong \mathrm{H}_*^{\mathbb{S}^1}(\mathscr{L}X; \mathbb{C}) . \tag{6.5.1}$$

Recall that $\mathscr{L}X$ is the space of all continuous maps $\mathbb{S}^1 \to X$ equipped with compact open topology, and as such, it carries a natural circle action (induced by the action of $\mathbb{S}^1$ on itself). Using the finite self-covering maps $\mathbb{S}^1 \to \mathbb{S}^1$, $e^{i\theta} \mapsto e^{in\theta}$, we may define, for all $n \geq 0$, the graded linear endomorphisms on the $\mathbb{S}^1$-equivariant homology of $\mathscr{L}X$:

$$\Phi_X^n : \mathrm{H}_*^{\mathbb{S}^1}(\mathscr{L}X; \mathbb{C}) \to \mathrm{H}_*^{\mathbb{S}^1}(\mathscr{L}X; \mathbb{C}) ,$$

which are called the *Frobenius operations*. Under the Jones isomorphism (6.5.1), the d-th Hodge summand $\mathrm{HC}_*^{(d)}(\mathcal{L}_X)$ of $\overline{\mathrm{HC}}_*(U\mathcal{L}_X)$ corresponds to the common eigenspace $\mathrm{H}_*^{\mathbb{S}^1, (d-1)}(\mathscr{L}X; \mathbb{C})$ of Frobenius operations Φ_X^n with the eigenvalues n^{d-1} for all $n \geq 0$ (see [26, Theorem 1.2]). On the other hand, by our Theorem 6.2.2, the representation homology of the Lie model $\mathcal{L}_X$ can be identified with the representation homology of X. Thus, for any simply connected space, the Drinfeld homomorphism (6.4.1) can be rewritten in purely topological terms

$$\mathrm{Sym}\big[\bigoplus_{i=1}^{l} \mathrm{H}_*^{\mathbb{S}^1,(m_i)}(\mathscr{L}X;\,\mathbb{C})\big] \;\to\; \mathrm{HR}_*(X,G)^G\,, \tag{6.5.2}$$

which exhibits an interesting relation between representation homology and $\mathbb{S}^1$-equivariant homology of simply connected spaces. Note that the domain and the target of the map (6.5.2) make sense for an arbitrary connected space X. It would be interesting to see if there is a direct construction of (6.5.2) that does not pass through the above identifications and works for all (not necessarily simply connected) spaces.

Now, for $X = \mathbb{CP}^r$, the result of Theorem 6.4.2 reads

$$\mathrm{HR}_*(\mathbb{CP}^r, G)^G \cong \mathrm{Sym}\big[\bigoplus_{i=1}^{l} \mathrm{H}_*^{\mathbb{S}^1,(m_i)}(\mathscr{L}(\mathbb{CP}^r);\mathbb{C})\big]\,. \tag{6.5.3}$$

Modulo Theorem 6.2.2, the isomorphism (6.5.3) is equivalent to the Strong Macdonald Conjecture. As we mentioned in the Introduction, this famous conjecture proposed by Feigin and Hanlon in the early 80s was recently proved in [60] by an algebraic *tour de force*. The above reformulation suggests that there might exist a more conceptual argument using topological means.

6.6 Lie models of non-simply connected spaces

In a series of recent papers [27, 28, 29], Buijs, Félix, Murillo and Tanré associated a free DG Lie algebra model $(\mathfrak{L}_X, d)$ to *any* finite simplicial complex X. Unlike Quillen models, the DG Lie algebras $\mathfrak{L}_X$ are assumed, in general, to be not connected but *complete* with respect to the canonical decreasing filtrarion $\mathfrak{L}^1 \supseteq \mathfrak{L}^2 \supseteq \ldots$ defined by $\mathfrak{L}^1 := \mathfrak{L}$ and $\mathfrak{L}^n := [\mathfrak{L}, \mathfrak{L}^{n-1}]$. The 0-simplices of X correspond to the degree -1 generators of $\mathfrak{L}_X$ that satisfy the Maurer-Cartan equation, while the n-simplices of X correspond to generators in degree $n-1$. For any connected, finite simplicial complex X, the DG Lie algebra $\mathfrak{L}_X$ itself is acyclic (i.e., $\mathrm{H}_*(\mathfrak{L}_X, d) = 0$). The topological information about X is contained in a DG Lie algebra $(\mathfrak{L}_X, d_v)$ obtained from $\mathfrak{L}_X$ by twisting its differential by Maurer-Cartan elements corresponding to the vertices of X, i.e. $d_v := d + [v\,,\, -\,]$, where v denotes (the Maurer-Cartan element corresponding to) a vertex of X. Now, the main result of [27] (see *loc. cit.*, Theorem A) says that, if X is simply connected, then $(\mathfrak{L}_X, d_v)$ is quasi-isomorphic to a Quillen model of X. This motivates the following conjectural generalization of our Theorem 6.2.2.

Let $(\mathfrak{L}_X, d)$ be a complete free DG Lie algebra model associated to a reduced simplicial set X. Let $d_v := d + [v\,,\,-]$ be the twisted differential on $\mathfrak{L}_X$ corresponding to the (unique) basepoint of X. Note that $\mathrm{HR}_0[(\mathfrak{L}_X, d_v), \mathfrak{g}]$ has a canonical augmentation ε corresponding to the trivial (zero) representation. Let $\widehat{\mathrm{HR}}_*[(\mathfrak{L}_X, d_v), \mathfrak{g}]$ denote the adic completion of $\mathrm{HR}_*[(\mathfrak{L}_X, d_v), \mathfrak{g}]$ with respect to the augmentation ideal of ε. Similarly, $\mathrm{HR}_0(X, G)$ has a canonical augmentation corresponding to the trivial (identity) representation of $\pi_1(X, v)$. Let $\widehat{\mathrm{HR}}_*(X, G)$ denote the corresponding completion of $\mathrm{HR}_*(X, G)$.

Conjecture 6.6.1. *There is an isomorphism of completed graded $\mathbb{Q}$-algebras*

$$\widehat{\mathrm{HR}}_*(X, G) \cong \widehat{\mathrm{HR}}_*[(\mathfrak{L}_X, d_v), \mathfrak{g}]\,.$$

Note that Conjecture 6.6.1 holds for X simply connected: indeed, in this case, $(\mathfrak{L}_X, d_v)$ is quasi-isomorphic to a Quillen model $\mathcal{L}_X$ of X and $\mathrm{HR}_0[(\mathfrak{L}_X, d_v), \mathfrak{g}] \cong \mathbb{Q}$. Thus, the right-hand side of the conjectured isomorphism is $\mathrm{HR}_*(\mathcal{L}_X, \mathfrak{g})$. Similarly, $\mathrm{HR}_0(X, G) = \mathbb{Q}$, which implies that $\widehat{\mathrm{HR}}_*(X, G) \cong \mathrm{HR}_*(X, G)$. Thus, Conjecture 6.6.1 is equivalent to Theorem 6.2.2 for simply connected spaces.

CHAPTER 7

EXAMPLES OF NON-SIMPLY CONNECTED SPACES

In this section, using standard topological decompositions, we compute representation homology of some classical non-simply connected spaces. Our examples include closed surfaces (both orientable and non-orientable) as well as some three-dimensional spaces (link complements in $\mathbb{R}^3$, lens spaces and general closed orientable 3-manifolds). The representation homology of surfaces and link complements is given in terms of classical Hochschild homology of $\mathcal{O}(G)$ (or $\mathcal{O}(G^n)$ for some $n \geq 2$) with twisted coefficients. The representation homology of a closed 3-manifold M is expressed in terms of a differential 'Tor', which gives rise to an (Eilenberg-Moore) spectral sequence converging to $\mathrm{HR}_*(M, G)$.

7.1 Surfaces

The torus

As a cell complex, the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ can be constructed as the homotopy cofibre (the mapping cone) of the map $\alpha : \mathbb{S}^1_c \to \mathbb{S}^1_a \vee \mathbb{S}^1_b$, where the subscripts on the circles indicate the generators of the respective fundamental groups, and the map itself is specified, up to homotopy, by its effect on these generators:

$$\alpha(c) = [a, b] := aba^{-1}b^{-1} \ . \tag{7.1.1}$$

Thus $\mathbb{T}^2 \simeq \mathrm{hocolim}[\, * \leftarrow \mathbb{S}^1_c \xrightarrow{\alpha} \mathbb{S}^1_a \vee \mathbb{S}^1_b]$, where the homotopy colimit is taken in the category $\mathtt{Top}_{0,*}$ of connected pointed spaces. Applying to this the Kan loop group functor $\mathbb{G}$ (more precisely, the composition[1] of $\mathbb{G}$ with the Eilenberg subcomplex functor, see Section 2.3), we get a simplicial group model for $\mathbb{T}^2$:

$$\mathbb{G}(\mathbb{T}^2) \cong \mathrm{hocolim}[\,1 \leftarrow \mathbb{F}_1 \stackrel{\alpha}{\to} \mathbb{F}_2\,]\,. \tag{7.1.2}$$

Here $\mathbb{F}_1$ and $\mathbb{F}_2$ are the free groups on the generators c and $\{a, b\}$ respectively; the map α is given by (7.1.1), and the homotopy colimit is taken in the category `sGr` of of simplicial groups.

Now, by Lemma 4.2.8, the derived representation functor DRep_G preserves homotopy pushouts for any algebraic group G. Hence, it follows from (7.1.2) that

$$\mathrm{DRep}_G[\mathbb{G}(\mathbb{T}^2)] \cong \mathrm{hocolim}[\,k \leftarrow \mathcal{O}(G) \xrightarrow{\alpha_*} \mathcal{O}(G \times G)\,]\,, \tag{7.1.3}$$

where the homotopy colimit is taken in $s\mathtt{CommAlg}_k$, and the map $\alpha_* : \mathcal{O}(G) \to \mathcal{O}(G \times G)$ is induced by (7.1.1) (explicitly, $\alpha_*(f)(x, y) = f([x, y])$ for $f \in \mathcal{O}(G)$). Since

$$\mathrm{hocolim}\,[\,k \leftarrow \mathcal{O}(G) \xrightarrow{\alpha_*} \mathcal{O}(G \times G)\,] \;\cong\; \mathcal{O}(G \times G) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k\,,$$

by Proposition 4.2.11, we conclude that

$$\mathrm{HR}_*(\mathbb{T}^2, G) \;\cong\; \mathrm{Tor}_*^{\mathcal{O}(G)}(\mathcal{O}(G \times G),\, k)\,, \tag{7.1.4}$$

where $\mathcal{O}(G \times G)$ is viewed as a (right) $\mathcal{O}(G)$-module via the algebra map α_*.

By standard homological algebra (see [42, Theorem 2.1, p. 185]), we can identify the Tor-groups in (7.1.4) as the classical Hochschild homology of $\mathcal{O}(G)$ with coefficients in the bimodule $\mathcal{O}(G \times G)$, where the right $\mathcal{O}(G)$-module structure is given via the map α_* and the left module structure via the augmentation map $\varepsilon : \mathcal{O}(G) \to k$:

$$\mathrm{HR}_*(\mathbb{T}^2, G) \;\cong\; \mathrm{HH}_*(\mathcal{O}(G),\, {}_{\varepsilon}\mathcal{O}(G \times G)_{\alpha})\,. \tag{7.1.5}$$

[1]Note that, being Quillen equivalences, both functors $\mathbb{G}$ and ES preserve homotopy colimits, and hence so does their composition.

Alternatively, for classical (matrix) groups G, we can give an explicit 'small' DG algebra model for the representation homology $\mathrm{HR}_*(\mathbb{T}^2, G)$. Specifically, let $\mathfrak{m} := \mathrm{Ker}(\varepsilon)$ denote the maximal (augmentation) ideal of $\mathcal{O}(G)$ corresponding to the identity element $e \in G$. Assume that $\mathfrak{m}$ is generated by a *regular* sequence of elements $(r_1, r_2, \ldots, r_d)$ in $\mathcal{O}(G)$, so that $d = \dim G$. Consider the free module $E := \mathcal{O}(G)^{\oplus d}$ and define the $\mathcal{O}$-module map $\pi : E \to \mathcal{O}(G)$ by $\pi(f_1, f_2, \ldots, f_d) := \sum_{i=1}^{d} r_i f_i$. Then, associated to (E, π) is the (global) Koszul complex $K_*(G) := (\Lambda^*_{\mathcal{O}(G)}(E), \delta_K)$ with differential

$$\delta_K(e_0 \wedge e_1 \wedge \ldots \wedge e_n) = \sum_{i=0}^{n} (-1)^i \, \pi(e_i) \, e_0 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_n \, .$$

Since $\mathfrak{m}$ is generated by a regular sequence, the canonical projection $K_*(G) \twoheadrightarrow \mathcal{O}(G)/\mathfrak{m} \cong k$ is a quasi-isomorphism of complexes, and therefore $K_*(G)$ is a free resolution of k over $\mathcal{O}(G)$. It follows from (7.1.4) that

$$\mathrm{Tor}_*^{\mathcal{O}(G)}(\mathcal{O}(G \times G), \, k) \, \cong \, \mathrm{H}_*[\mathcal{A}(\mathbb{T}^2, G)] \, , \tag{7.1.6}$$

where $\mathcal{A}(\mathbb{T}^2, G) := \mathcal{O}(G \times G) \otimes_{\mathcal{O}(G)} K_*(G)$ is a commutative DG algebra with differential $d = \mathrm{id} \otimes \delta_K$. In particular, $\mathrm{HR}_i(\mathbb{T}^2, G) = 0$ for all $i > \dim G$.

We conclude this example with a conjectural description of the G-invariant part of representation homology $\mathrm{HR}_*(\mathbb{T}^2, G)^G$. Our conjecture can be viewed as a multiplicative analogue of the derived Harish-Chandra conjecture proposed in [24].

Assume that G is a reductive algebraic group of rank $l \geq 1$ defined over an algebraically closed field k of characteristic zero. Let $T \subset G$ be a Cartan subgroup (i.e., a maximal torus) in G, and let W be the corresponding Weyl group. Note that, since T is commutative, the map $\alpha_* : \mathcal{O}(T) \to \mathcal{O}(T \times T)$ associated to T factors through the augmentation $\varepsilon : \mathcal{O}(T) \to k$. Hence, by

(7.1.4), we have canonical isomorphisms

$$\begin{aligned} \mathrm{HR}_*(\mathbb{T}^2, T) &\cong \mathrm{Tor}_*^{\mathcal{O}(T)}(\mathcal{O}(T\times T),\, k) \\ &\cong \mathcal{O}(T\times T)\otimes \mathrm{Tor}_*^{\mathcal{O}(T)}(k,\, k) \\ &\cong \mathcal{O}(T\times T)\otimes \Lambda_k^*(\mathfrak{m}_T/\mathfrak{m}_T^2) \\ &\cong \mathcal{O}(T\times T)\otimes \Lambda_k^*(\mathfrak{h}^*)\,, \end{aligned} \tag{7.1.7}$$

where $\mathfrak{m}_T$ is the augmentation ideal for T and $\mathfrak{h} = (\mathfrak{m}_T/\mathfrak{m}_T^2)^*$ is the Lie algebra of T (i.e., a Cartan subalgebra of $\mathfrak{g}$).

Now, by functoriality, the natural inclusion $T \hookrightarrow G$ induces a map of simplicial commutative algebras

$$\Phi_G(\mathbb{T}^2):\ \mathrm{DRep}_G[\mathbb{G}(\mathbb{T}^2)]^G \to \mathrm{DRep}_T[\mathbb{G}(\mathbb{T}^2)]^W\,, \tag{7.1.8}$$

which is (a multiplicative analogue of) the derived Harish-Chandra homomorphism constructed in [24]. Then, the multiplicative version of the derived Harish-Chandra conjecture states

Conjecture 7.1.9. *Assume that G is one of the classical groups $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, $\mathrm{Sp}_{2n}(k)$, $n \geq 1$, or any simply-connected, semi-simple*[2] *affine algebraic group. Then the derived Harish-Chandra homomorphism* (7.1.8) *is a weak equivalence in $s\mathtt{CommAlg}_k$. Hence, by* (7.1.7), *there is an isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(\mathbb{T}^2, G)^G \cong [\mathcal{O}(T\times T)\otimes \Lambda_k^*(\mathfrak{h}^*)]^W\,. \tag{7.1.10}$$

We illustrate Conjecture 7.1.9 for $G = \mathrm{GL}_n$. Since $\mathcal{O}(\mathrm{GL}_n) \cong k[x_{ij}, \det(x_{ij})^{-1}]_{1\le i,j\le n}$, the elements $\{x_{ij} - \delta_{ij}\}_{1\le i,j\le n}$ form a regular sequence

[2] It is known that every simply-connected reductive affine algebraic group is automatically semi-simple. This follows from two classical facts: (1) every reductive Lie algebra is a product of a semi-simple one and an abelian one; (2) there are no nontrivial simply-connected abelian reductive algebraic groups.

in $\mathcal{O}(\mathrm{GL}_n)$ generating the maximal ideal $\mathfrak{m}$, so we have a canonical commutative DG algebra representing $\mathrm{HR}_*(\mathbb{T}^2, \mathrm{GL}_n)$:

$$\mathcal{A}(\mathbb{T}^2, \mathrm{GL}_n) \cong k[x_{ij},\, y_{ij},\, \theta_{ij};\, \det(X)^{-1},\, \det(Y)^{-1}]_{1\le i,j\le n}\ .$$

Here the variables x_{ij} and y_{ij} have homological degree 0, θ_{ij} have homological degree 1, and $\det(X)$ and $\det(Y)$ denote the determinants of the generic matrices $X := \|x_{ij}\|$ and $Y := \|y_{ij}\|$. The differential on $\mathcal{A}(\mathbb{T}^2, \mathrm{GL}_n)$ can be written in matrix terms as

$$d\,\Theta = XYX^{-1}Y^{-1} - I_n\ ,$$

where $\Theta := \|\theta_{ij}\|$ and I_n is the identity $n \times n$-matrix. The Harish-Chandra homomorphism

$$\Phi_{\mathrm{GL}_n}(\mathbb{T}^2):\ \mathcal{A}(\mathbb{T}^2, \mathrm{GL}_n)^{\mathrm{GL}_n} \to k\left[x_1^{\pm 1},\, \ldots,\, x_n^{\pm 1},\, y_1^{\pm 1}, \ldots,\, y_n^{\pm 1},\, \theta_1,\, \ldots,\, \theta_n\right]^{S_n}$$

is given explicitly (on generators) by the following map

$$x_{ij} \mapsto \delta_{ij} x_i\ , \quad y_{ij} \mapsto \delta_{ij} y_i\ , \quad \theta_{ij} \mapsto \delta_{ij}\theta_i\ ,$$

and the derived Harish-Chandra conjecture asserts that $\Phi_{\mathrm{GL}_n}(\mathbb{T}^2)$ induces an isomorphism (*cf.* (7.1.10))

$$\mathrm{HR}_*(\mathbb{T}^2, \mathrm{GL}_n)^{\mathrm{GL}_n} \xrightarrow{\sim} k\left[x_1^{\pm 1},\, \ldots,\, x_n^{\pm 1},\, y_1^{\pm 1}, \ldots,\, y_n^{\pm 1},\, \theta_1,\, \ldots,\, \theta_n\right]^{S_n}\ , \qquad (7.1.11)$$

where $\theta_1, \ldots, \theta_n$ have homological degree 1 and the symmetric group S_n acts diagonally by permuting the variables. Note that, in the case of $\mathrm{GL}_n(k)$, unlike for other algebraic groups, Conjecture 7.1.9 follows from the derived Harish-Chandra conjecture for the corresponding Lie algebra $\mathfrak{gl}_n(k)$ stated in [24]. This is because the Harish-Chandra map $\Phi_{\mathrm{GL}_n}(\mathbb{T}^2)$ can be obtained by formally localizing the derived Harish-Chandra map for the Lie algebra $\mathfrak{gl}_n(k)$ (*cf.* [24, Sect. 4]). In particular, the evidence collected in [24] for $\mathfrak{gl}_n(k)$ also supports Conjecture 7.1.9 for $\mathrm{GL}_n(k)$. we list some of this evidence here.

1) Conjecture 7.1.9 holds for $\mathrm{GL}_2(k)$ and $\mathrm{GL}_\infty(k)$. This follows from Theorem 4.1 and Theorem 4.2(ii) of [24].

2) For all $n \geq 1$, the map (7.1.11) is degreewise surjective. This follows from [24, Theorem 4.2(i)].

3) For all $n \geq 1$, $\mathrm{HR}_i(\mathbb{T}^2, \mathrm{GL}_n)^{\mathrm{GL}_n} = 0$ for $i > n$. This follows from [22, Theorem 27].

4) For any G as in Conjecture 7.1.9, the map (7.1.11) is an isomorphism in homological degree *zero*, i.e. $\mathrm{HR}_0(\mathbb{T}^2, G)^G \cong \mathcal{O}(T \times T)^W$. This follows from a theorem of Thaddeus [159] (see also [148]).

Finally, we remark that, for $G = \mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$ and $\mathrm{Sp}_{2n}(k)$, the Harish-Chandra map is known to be an isomorphism in homological degree 0: $\mathrm{HR}_0(\mathbb{T}^N, G)^G \cong \mathcal{O}(T^N)^W$ for all tori $\mathbb{T}^N$, $N \geq 2$ (see [148]). However, by results of [24, Sect. 5.2], the above isomorphism does not extend to higher homological degrees when $N \geq 3$. In other words, the derived Harish-Chandra homomorphism $\Phi_{\mathrm{GL}_n}(\mathbb{T}^N)$ is not a week equivalence for higher dimensional tori $\mathbb{T}^N$, $N \geq 3$.

Riemann surfaces

The above computation of representation homology of the 2-torus naturally generalizes to Riemann surfaces of an arbitrary genus. To be precise, let Σ_g denote a closed connected orientable surface of genus $g \geq 1$. As a 2-dimensional cell complex, Σ_g can be described as the homotopy cofibre of the map $\alpha^g : \mathbb{S}^1_c \to \vee_{i=1}^g \left(\mathbb{S}^1_{a_i} \vee \mathbb{S}^1_{b_i}\right)$ defined by

$$\alpha^g(c) = [a_1, b_1]\,[a_2, b_2]\, \ldots\, [a_g, b_g]\,, \tag{7.1.12}$$

where $a_1, b_1, \ldots, a_g, b_g$ denote the a- and b-cycles on Σ_g generating the fundamental group $\pi_1(\Sigma_g, *)$. This gives the simplicial group model $\mathbb{G}(\Sigma_g) \cong \mathrm{hocolim}[\, 1 \leftarrow \mathbb{F}_1 \xrightarrow{\alpha^g} \mathbb{F}_{2g}\,]$ of Σ_g, which, in turn, implies

$$\mathrm{DRep}_G[\mathbb{G}(\Sigma_g)] \cong \mathrm{hocolim}\,[\, k \leftarrow \mathcal{O}(G) \xrightarrow{\alpha^g_*} \mathcal{O}(G^{2g})\,] \;\cong\; \mathcal{O}(G^{2g}) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k\ ,$$

where the map $\alpha^g_* : \mathcal{O}(G) \to \mathcal{O}(G^{2g})$ is defined by

$$\alpha^g_*(f)(x_1, y_1, \ldots, x_g, y_g) := f([x_1, y_1]\,[x_2, y_2]\,\ldots\,[x_g, y_g])\ , \quad f \in \mathcal{O}(G)\ .$$

By Proposition 4.2.11, we conclude

$$\mathrm{HR}_*(\Sigma_g, G) \;\cong\; \mathrm{Tor}^{\mathcal{O}(G)}_*(\mathcal{O}(G^{2g}),\, k) \;\cong\; \mathrm{HH}_*(\mathcal{O}(G),\, {}_{\varepsilon}\mathcal{O}(G^{2g})_{\alpha}) \tag{7.1.13}$$

where ${}_{\varepsilon}\mathcal{O}(G^{2g})_{\alpha}$ is the bimodule with left and and right $\mathcal{O}(G)$-module structure given by the maps ε and α^g_*, respectively.

In case when $\mathfrak{m} \subset \mathcal{O}(G)$ is generated by a regular sequence, we can also express the representation homology of Σ_g as the homology of the commutative DG algebra $\mathcal{A}(\Sigma_g, G) := \mathcal{O}(G^{2g}) \otimes_{\mathcal{O}(G)} K_*(G)$, where $K_*(G)$ is the global Koszul complex constructed in Section 7.1:

$$\mathrm{HR}_*(\Sigma_g, G) \;\cong\; \mathrm{H}_*[\,\mathcal{A}(\Sigma_g, G)\,]\ .$$

As in the torus case, for a reductive group G with a Cartan subgroup T, there is an algebra map induced by the derived Harish-Chandra homomorpism $\Phi_G(\Sigma_G)$:

$$\mathrm{HR}_*(\Sigma_g, G)^G \;\to\; [\mathcal{O}(T^{2g}) \otimes \Lambda^*_k(\mathfrak{h}^*)]^W$$

where W operates diagonally on the target. However, for $g \geq 2$, this map seems far from being an isomorphism even for classical groups G as in Conjecture 7.1.9.

Non-orientable surfaces

Let $\mathcal{N}_g$ denote a closed connected non-orientable surface of genus g. For example, $\mathcal{N}_1 \cong \mathbb{RP}^2$ and $\mathcal{N}_2$ is the Klein bottle. It is known that $\mathcal{N}_g$ is homotopy equivalent to the homotopy cofibre of the map $\beta^g : \mathbb{S}^1_c \to \vee_{i=1}^{g} \mathbb{S}^1_{a_i}$:

$$\beta^g(c) = a_1^2\, a_2^2\, \ldots\, a_g^2\,, \tag{7.1.14}$$

where $c, a_1, \ldots, a_g$ denote generators of the corresponding fundamental groups. This gives the following simplicial group model for $\mathcal{N}_g$:

$$\mathbb{G}(\mathcal{N}_g) \cong \mathrm{hocolim}[\, 1 \leftarrow \mathbb{F}_1 \xrightarrow{\beta^g} \mathbb{F}_g \,]\,,$$

which, in turn, gives the presentation

$$\mathrm{DRep}_G[\mathbb{G}(\mathcal{N}_g)] \cong \mathrm{hocolim}\,[\, k \leftarrow \mathcal{O}(G) \xrightarrow{\beta^g_*} \mathcal{O}(G^g)\,] \;\cong\; \mathcal{O}(G^g) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k\,,$$

where $\beta^g_* : \mathcal{O}(G) \to \mathcal{O}(G^g)$ is defined by $\beta^g_*(f)(x_1, \ldots, x_g) = f(x_1^2\, x_2^2\, \ldots\, x_g^2)$ for $f \in \mathcal{O}(G)$. Thus,

$$\mathrm{HR}_*(\mathcal{N}_g, G) \;\cong\; \mathrm{Tor}_*^{\mathcal{O}(G)}(\mathcal{O}(G^g),\, k) \;\cong\; \mathrm{HH}_*(\mathcal{O}(G),\, {}_{\varepsilon}\mathcal{O}(G^g)_{\beta})\,.$$

7.2 3-Manifolds

Link complements in $\mathbb{R}^3$

By a link L in $\mathbb{R}^3$ we mean a smooth (oriented) embedding of the disjoint union $\mathbb{S}^1 \sqcup \ldots \sqcup \mathbb{S}^1$ of (a finite number of) copies of $\mathbb{S}^1$ into $\mathbb{R}^3$. The link complement $X := \mathbb{R}^3 \setminus L$ is then defined to be the complement of an (open) tubular neighborhood of the image of L in $\mathbb{R}^3$. To describe a simplicial group model for X we recall two classical facts from geometric topology (*cf.* [12]). First, by a well-known theorem of J. W. Alexander, every link L in $\mathbb{R}^3$ can be obtained geometrically as the closure of a braid β in $\mathbb{R}^3$ (we write $L = \hat{\beta}$ to indicate this relation). Second, for each $n \geq 1$, the braids on n strands in $\mathbb{R}^3$ form a group B_n (the Artin braid group), which admits a faithful representation by automorphisms of the free group $\mathbb{F}_n$ (the Artin representation). Specifically, the group B_n is generated by $n-1$ elements ("flips") $\sigma_1,\, \sigma_2,\, \ldots,\, \sigma_{n-1}$ subject to the relations

$$\sigma_i\,\sigma_j = \sigma_j\,\sigma_i \quad (\text{if } |i-j| > 1)\,, \qquad \sigma_i\,\sigma_j\,\sigma_i = \sigma_j\,\sigma_i\,\sigma_j \quad (\text{if } |i-j| = 1)\,,$$

and in terms of these generators, the Artin representation $B_n \to \mathrm{Aut}(\mathbb{F}_n)$ is given by

$$\sigma_i \;:\; \begin{cases} x_i & \mapsto \quad x_i\,x_{i+1}\,x_i^{-1} \\ x_{i+1} & \mapsto \quad x_i \\ x_j & \mapsto \quad x_j \quad (j \neq i, i+1) \end{cases} \tag{7.2.1}$$

To simplify the notation we will identify B_n with its image in $\mathrm{Aut}(\mathbb{F}_n)$ under (7.2.1).

The next proposition can be viewed as a refinement of a classical theorem of Artin and Birman [12, Theorem 2.2] describing the fundamental group of the link complement $\mathbb{R}^3 \backslash L$ in terms of the Artin representation (see Remark below).

Proposition 7.2.2. *Let $L = \hat{\beta}$ be a link in $\mathbb{R}^3$ given by the closure of a braid $\beta \in B_n$. Then*

$$\mathbb{G}(\mathbb{R}^3 \backslash L) \;\cong\; \mathrm{hocolim}\,[\,\mathbb{F}_n \xleftarrow{(\beta,\,\mathrm{id})} \mathbb{F}_n \amalg \mathbb{F}_n \xrightarrow{(\mathrm{id},\,\mathrm{id})} \mathbb{F}_n\,]\,, \tag{7.2.3}$$

where β acts on $\mathbb{F}_n$ via the Artin representation (7.2.1).

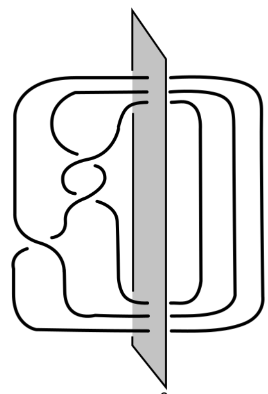
Remark 7.2.4. Note that the homotopy pushout in (7.2.3) coincides with the homotopy coequalizer $\boldsymbol{L}\mathrm{coeq}[\,\mathbb{F}_n \underset{\beta}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{F}_n\,]$ of the two endomorphisms id and β

of $\mathbb{F}_n$. Hence, (7.2.3) implies

$$\pi_1(\mathbb{R}^3 \backslash L, *) \cong \pi_0[\mathbb{G}(\mathbb{R}^3 \backslash L)] \cong \operatorname{coeq}\big[\, \mathbb{F}_n \overset{\mathrm{id}}{\underset{\beta}{\rightrightarrows}} \mathbb{F}_n \,\big] \cong \langle x_1, \ldots, x_n \,|\, \beta(x_1) = x_1, \ldots, \beta(x_n) = x_n \rangle \,,$$

which is the Artin-Birman presentation of the link group $\pi(L) := \pi_1(\mathbb{R}^3 \backslash L, *)$.

Proof. The proof is based on a simple van Kampen type argument (*cf.* [12]). Let's place the n-braid β in a regular position in the region $x < 0$ in $\mathbb{R}^3$, so that its starting points $\{p_1, p_2, \ldots, p_n\}$ and end points $\{q_1, q_2, \ldots, q_n\}$ are located on the z-axis with coordinates $q_1 < q_2 < \ldots < q_n < p_n < p_{n-1} < \ldots < p_1$. The link L is the closure of β obtained by joining the points p_i to q_i ($i = 1, 2, \ldots, n$) by simple arcs in the region $x > 0$, as shown in the picture



Now, let $X := \mathbb{R}^3 \backslash L$ denote the complement of L. Define

$$X_{\geq 0} := \{(x, y, z) \in X \,:\, x \geq 0\}\,, \quad X_{\leq 0} := \{(x, y, z) \in X \,:\, x \leq 0\}\,, \quad X_0 := X_{\geq 0} \cap X_{\leq 0}\,.$$

with a (common) basepoint $*$ in X_0. It is easy to see that $X_{\geq 0}$ is homeomorphic to the cylinder over $\mathbb{R}^2 \backslash \{p_1, \ldots, p_n\}$, which is, in turn, homotopic to $\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}$, where $\mathbb{D}^2$ is a two-dimensional disk in (the yz-plane) $\mathbb{R}^2$ encompassing the points $\{p_1, \ldots, p_n\}$. Similarly, we have $X_{\leq 0} \cong (\mathbb{R}^2 \setminus \{q_1, \ldots, q_n\}) \times [0, 1] \simeq \mathbb{D}^2 \backslash \{q_1, \ldots, q_n\}$, and

$$X_0 \simeq \mathbb{D}^2 \backslash \{p_1, \ldots, p_n, q_1, \ldots, q_n\} \simeq \mathbb{D}^2 \backslash \{p_1, \ldots, p_n\} \vee \mathbb{D}^2 \backslash \{q_1, \ldots, q_n\}\,.$$

Under these identifications, the natural inclusions $X_{\leq 0} \hookleftarrow X_0 \hookrightarrow X_{\geq 0}$ can be identified with

$$\mathbb{D}^2 \backslash \{q_1, \ldots, q_n\} \xleftarrow{(f_\beta, \mathrm{id})} \mathbb{D}^2 \backslash \{p_1, \ldots, p_n\} \vee \mathbb{D}^2 \backslash \{q_1, \ldots, q_n\} \xrightarrow{(\mathrm{id}, f_e)} \mathbb{D}^2 \backslash \{p_1, \ldots, p_n\} \,, \tag{7.2.5}$$

where the map f_β is determined (uniquely up to homotopy) by the braid β and the map f_e is determined by the trivial braid connecting the points p_i and q_i. Thus, we can represent X in $\mathrm{Ho}(\mathtt{Top}_{0,*})$ as the homotopy pushout of the diagram (7.2.5).

Next, recall that B_n can be identified with the mapping class group of $\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}$ comprising (the isotopy classes of) orientation-preserving homeomorphisms that fix pointwise the boundary of $\mathbb{D}^2$. As a mapping class group, B_n acts naturally on the fundamental group $\pi_1(\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}, *)$ and the latter can be identified with the free group $\mathbb{F}_n$ on generators $x_1, \ldots, x_n$ represented by small loops in $\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}$ around the points p_i. It is well-known (see [12]) that the action of B_n on $\mathbb{F}_n$ arising from this construction is precisely the Artin representation (7.2.1). Now, using the map f_e we identify $\mathbb{D}^2 \backslash \{q_1, \ldots, q_n\}$ with $\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}$ in (7.2.5) and apply the loop group functor to this diagram of spaces. As a result, we get the equivalence (7.2.3), which completes the proof of the proposition. □

To state our main theorem we introduce some notation. First, observe that, for any algebraic group G, the Artin representation $B_n \hookrightarrow \mathrm{Aut}(\mathbb{F}_n)$ induces naturally a braid group action $B_n \to \mathrm{Aut}[\mathcal{O}(G^n)]$, which we denote by $\beta \mapsto \beta_*$. On the standard generators, this action is defined by

$$(\sigma_i)_* : \mathcal{O}(G^n) \to \mathcal{O}(G^n) \,, \quad f(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) \mapsto f(g_1, \ldots, g_i g_{i+1} g_i^{-1}, g_i, \ldots, g_n) \,.$$

Now, for a braid $\beta \in B_n$, we let $O(G^n)_\beta$ denote the $\mathcal{O}(G^n)$-bimodule whose underlying vector space is $\mathcal{O}(G^n) = \mathcal{O}(G)^{\otimes n}$, the left action of $\mathcal{O}(G^n)$ is given by multiplication, while the right action is twisted by the automorphism β_*.

Theorem 7.2.6. *Let $L = \hat{\beta}$ be a link in $\mathbb{R}^3$ given by the closure of a braid $\beta \in B_n$. Then*

$$\mathrm{DRep}_G[\mathbb{G}(\mathbb{R}^3 \backslash L)] \cong \mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta \ .$$

Consequently,

$$\mathrm{HR}_*(\mathbb{R}^3 \backslash L, G) \ \cong \ \mathrm{HH}_*(\mathcal{O}(G^n),\, \mathcal{O}(G^n)_\beta) \ . \tag{7.2.7}$$

Proof. By Proposition 7.2.2 and Lemma 4.2.8, we have

$$\begin{aligned}
\mathrm{DRep}_G[\mathbb{G}(\mathbb{R}^3 \backslash L)] \ &\cong \ \mathrm{hocolim}\,[\,\mathcal{O}(G^n) \xleftarrow{(\beta_*,\,\mathrm{id})} \mathcal{O}(G^n) \otimes_k \mathcal{O}(G^n) \xrightarrow{(\mathrm{id},\,\mathrm{id})} \mathcal{O}(G^n)\,] \\
&\cong \ \mathrm{hocolim}\,[\,\mathcal{O}(G^n) \xleftarrow{(\beta_*,\,\mathrm{id})} \mathcal{O}(G^{2n}) \xrightarrow{(\mathrm{id},\,\mathrm{id})} \mathcal{O}(G^n)\,] \\
&\cong \ \mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta \ .
\end{aligned}$$

This completes the proof of the theorem. □

Remark 7.2.8. Theorem 7.2.6 exhibits an interesting analogy between representation homology of link complements in $\mathbb{R}^3$ and their contact homology in the sense of L. Ng [120, 121]. This analogy is explained in the recent paper [11], where a new algebraic construction of knot contact homology is given. Roughly speaking, in terminology of [11], $\mathrm{DRep}_G[\mathbb{G}(\mathbb{R}^3 \backslash L)]$ represents the algebraic 'homotopy closure' of the braid $\beta \in B_n$ in the category of simplicial commutative algebras, while the knot contact homology of L represents the homotopy braid closure of $\beta \in B_n$ in the category of (small pointed) DG categories.

Link complements in $\mathbb{S}^3$

Note that Theorem 7.2.6 computes the representation homology of the topological *space* $\mathbb{R}^3 \backslash L$, not of the *link group* $\pi(L)$, which is the fundamental group of $\mathbb{R}^3 \backslash L$. Even when L is a knot in $\mathbb{R}^3$ (i.e., a link with one component), the representation homologies $\mathrm{HR}_*(\mathbb{R}^3 \backslash L, G)$ and $\mathrm{HR}_*(\pi(L), G)$ differ, because $\mathbb{R}^3 \backslash L$ is not a $K(\pi, 1)$-space[3]. In knot theory, one is usually interested in representation varieties of the knot group $\pi(L)$, so it is important to understand the relation between $\mathrm{HR}_*(\mathbb{R}^3 \backslash L, G)$ and $\mathrm{HR}_*(\pi(L), G)$. A natural way to approach this problem is to consider L as a link in $\mathbb{S}^3$ by adding to $\mathbb{R}^3$ one point at infinity. If $L \subset \mathbb{R}^3 \subset \mathbb{S}^3$ is a knot, by Papakyriakopoulos' Sphere Theorem, the complement $\mathbb{S}^3 \backslash L$ is an aspherical space, and $\pi_1(\mathbb{S}^3 \backslash L, *) \cong \pi_1(\mathbb{R}^3 \backslash L, *) = \pi(L)$. Hence, for any knot L, $\mathrm{HR}_*(\pi(L), G) \cong \mathrm{HR}_*(\mathbb{S}^3 \backslash L, G)$, so it suffices to clarify the relation between $\mathrm{HR}_*(\mathbb{R}^3 \backslash L, G)$ and $\mathrm{HR}_*(\mathbb{S}^3 \backslash L, G)$.

To this end, we observe that the natural inclusion $\mathbb{R}^3 \backslash L \hookrightarrow \mathbb{S}^3 \backslash L$ fits into the cofibration sequence $\mathbb{S}^2 \overset{i}{\hookrightarrow} \mathbb{R}^3 \backslash L \hookrightarrow \mathbb{S}^3 \backslash L$, so that

$$\mathbb{S}^3 \backslash L \cong \mathrm{hocolim}\,[\, * \leftarrow \mathbb{S}^2 \xrightarrow{i} \mathbb{R}^3 \backslash L\,]\,, \tag{7.2.9}$$

where $\mathbb{S}^2 \subset \mathbb{R}^3$ is chosen in such a way that it encloses L in $\mathbb{R}^3$. Applying the Kan functor to (7.2.9), we get

$$\mathbb{G}(\mathbb{S}^3 \backslash L) \cong \mathrm{hocolim}\,[\, 1 \leftarrow \mathbb{G}(\mathbb{S}^2) \xrightarrow{i_*} \mathbb{G}(\mathbb{R}^3 \backslash L)\,]\,. \tag{7.2.10}$$

To describe the induced map i_*, we note that $\mathbb{S}^2 \cong \Sigma\, \mathbb{S}^1 \cong \mathrm{hocolim}\,[\, * \leftarrow \mathbb{S}^1 \to *\,]$, hence

$$\mathbb{G}(\mathbb{S}^2) \cong \mathrm{hocolim}\,[\, 1 \leftarrow \mathbb{F}_1 \to 1\,]\,. \tag{7.2.11}$$

[3]In fact, $\pi_2(\mathbb{R}^3 \backslash L, *) \neq 0$.

Now, if we identify $\mathbb{G}(\mathbb{R}^3 \backslash L)$ as in Proposition 7.2.2, then i_* is determined by the morphism of diagrams

$$
\begin{tikzcd}
1 & \mathbb{F}_1 \arrow[l] \arrow[r] & 1 \\
\mathbb{F}_n & \mathbb{F}_n \amalg \mathbb{F}_n \arrow[l, "(\beta, \mathrm{id})"'] \arrow[r, "(\mathrm{id}, \mathrm{id})"] & \mathbb{F}_n
\end{tikzcd}
\tag{7.2.12}
$$

where the map in the middle is given (on free generators) by $x \mapsto (x_1\, x_2\, \ldots\, x_n)\,(y_1\, y_2\, \ldots\, y_n)^{-1}$. Note that the left square in (7.2.12) commutes because the product $x_1 x_2 \ldots x_n \in \mathbb{F}_n$ stays fixed under the Artin representation for any $\beta \in B_n$.

The map $i_* : \mathbb{G}(\mathbb{S}^2) \to \mathbb{G}(\mathbb{R}^3 \setminus L)$ induces a map of simplicial commutative algebras

$$
i_* : \ \mathrm{DRep}_G[\mathbb{G}(\mathbb{S}^2)] \to \mathrm{DRep}_G[\mathbb{G}(\mathbb{R}^3 \setminus L)] \,, \tag{7.2.13}
$$

which (to simplify the notation) we denote by the same symbol. By Theorem 7.2.6,

$$
\mathrm{DRep}_G[\mathbb{G}(\mathbb{R}^3 \setminus L)] \;\cong\; \mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta \ .
$$

On the other hand, by (7.2.11),

$$
\mathrm{DRep}_G[\mathbb{G}(\mathbb{S}^2)] \;\cong\; k \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k \;\cong\; \Lambda_k(\mathfrak{m}/\mathfrak{m}^2) \cong \Lambda_k(\mathfrak{g}^*) \ ,
$$

where the exterior algebra $\Lambda_k(\mathfrak{g}^*) := \mathrm{Sym}_k(\mathfrak{g}^*[1])$ is regarded as a DG algebra with trivial differential. With these identifications, the map (7.2.13) is induced by the algebra homomorphism

$$
\mathcal{O}(G) \to \mathcal{O}(G^{2n}) \ , \quad f(x) \mapsto f((x_1\, x_2\, \ldots\, x_n)\,(y_1\, y_2\, \ldots\, y_n)^{-1}) \ . \tag{7.2.14}
$$

Now, we can regard $\mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta$ as a DG module over the DG algebra $k \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k \cong \Lambda(\mathfrak{g}^*)$. As a consequence of (7.2.10) and Theorem 7.2.6, we have then the following

Theorem 7.2.15. *Let $L = \hat{\beta}$ be a link in $\mathbb{S}^3$ given by the closure of a braid $\beta \in B_n$. Then*

$$\mathrm{DRep}_G[\mathbb{G}(\mathbb{S}^3\backslash L)] \;\cong\; k \otimes^{\boldsymbol{L}}_{\Lambda(\mathfrak{g}^*)} [\mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta] \,.$$

Consequenlty, there is a natural spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{\Lambda(\mathfrak{g}^*)}_*(k,\, \mathrm{HH}_*(\mathcal{O}(G^n),\, \mathcal{O}(G^n)_\beta) \implies \mathrm{HR}_*(\mathbb{S}^3\backslash L,\, G)\,,$$

converging to the representation homology of $\mathbb{S}^3\backslash L$.

Lens spaces

Recall that, for coprime integers p and q, the lens space $L(p,q)$ of type (p,q) is defined as the quotient $\mathbb{S}^3/\mathbb{Z}_p$ of the 3-sphere $\mathbb{S}^3$ viewed as the unit sphere in $\mathbb{C}^2$ modulo the (free) action of the cyclic group $\mathbb{Z}_p$ given by $(z,w) \mapsto (e^{2\pi i/p}\, z,\, e^{2\pi i q/p}\, w)$. This definition shows that $L(p,q)$ is a compact connected 3-manifold, whose universal cover is $\mathbb{S}^3$ and the fundamental group is $\mathbb{Z}_p$. Special cases include $L(1,0) \cong \mathbb{S}^3$, $L(0,1) \cong \mathbb{S}^1 \times \mathbb{S}^2$ and $L(2,1) \cong \mathbb{RP}^3$.

To compute the representation homology of $L(p,q)$ we will use a well-known topological construction of these spaces in terms of Dehn surgery in $\mathbb{S}^3$ (see, e.g., [139, Chap. 3B]). Recall that if $K \subset \mathbb{S}^3$ is a knot in $\mathbb{S}^3$ and p, q are two integer numbers, the p/q Dehn surgery on K is a 3-dimensional space obtained by removing from $\mathbb{S}^3$ the interior $\mathring{N}(K)$ of a regular tubular neighborhood $N(K)$, which is a 3-dimensional solid torus $\mathbb{S}^1 \times \mathbb{D}^2$, and then gluing $\mathbb{S}^1 \times \mathbb{D}^2$ back to $\mathbb{S}^3\backslash\mathring{N}(K)$ in such a way that the meridional curve of $\mathbb{S}^1 \times \mathbb{D}^2$ is identified with a (p,q)-curve on the boundary of $\mathbb{S}^3\backslash\mathring{N}(K)$. For the trivial knot $K \subset \mathbb{S}^3$, it is easy to see that the p/q Dehn surgery on K gives precisely the lens space $L(p,q)$. In this case, the knot complement $\mathbb{S}^3\backslash\mathring{N}(K)$ is homeomorphic to the solid torus

$\mathbb{S}^1 \times \mathbb{D}^2$, so the space $L(p,q)$ can be obtained by gluing together two solid tori along their boundary.

To describe this in more precise terms, we consider the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ as a subset in $\mathbb{C}^2$:

$$\mathbb{S}^1 \times \mathbb{D}^2 = \{(z,w) \in \mathbb{C}^2 \,:\, |z| = 1\,,\ |w| \le 1\}\,.$$

We identify $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as the boundary of $\mathbb{S}^1 \times \mathbb{D}^2$ in $\mathbb{C}^2$ and denote by $i : \mathbb{T}^2 \hookrightarrow \mathbb{S}^1 \times \mathbb{D}^2$ the natural inclusion.

Now, for the given pair (p,q) of coprime numbers, we choose $m, n \in \mathbb{Z}$, so that $mq - np = 1$, and define the 'gluing' map $\gamma :\ \mathbb{T}^2 \to \mathbb{S}^1 \times \mathbb{D}^2$ by[4]

$$\gamma(z,\ w) := (z^m w^p,\ z^n w^q)\,. \tag{7.2.16}$$

Then the p/q Dehn surgery construction of $L(p,q)$ can be described as the pushout in $\mathtt{Top}_{0,*}$:

$$L(p,q) \cong \mathrm{colim}\,[\,\mathbb{S}^1 \times \mathbb{D}^2 \xleftarrow{i} \mathbb{T}^2 \xrightarrow{\gamma} \mathbb{S}^1 \times \mathbb{D}^2\,]\,. \tag{7.2.17}$$

Since i is a cofibration in $\mathtt{Top}_{0,*}$, we can replace the colimit in (7.2.17) by a homotopy colimit and then replace the diagram of solid tori by a homotopy equivalent diagram of circles:

$$L(p,q) \;\cong\; \mathrm{hocolim}\,[\,\mathbb{S}^1 \xleftarrow{\pi} \mathbb{T}^2 \xrightarrow{\bar{\gamma}} \mathbb{S}^1\,]\,. \tag{7.2.18}$$

In this diagram, the map π is given by the canonical projection $(z,w) \mapsto z$ and $\bar{\gamma}$ is the composition $\pi \circ \gamma$ defined by $(z,w) \mapsto z^m w^p$. Now, applying the Kan loop

[4]Note that the map γ factors as $\gamma = i \circ \gamma_A$, where $\gamma_A : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism of the torus defined by formula (7.2.16) that depends on the matrix $A = \begin{pmatrix} m & p \\ n & q \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. The assignment $A \mapsto \gamma_A$ induces an isomorphism from $\mathrm{SL}_2(\mathbb{Z})$ to the mapping class group of $\mathbb{T}^2$. Formula (7.2.18) shows that $L(p.q)$ can be obtained as a join of two circles, $\mathbb{S}^1 \star_\gamma \mathbb{S}^1$, twisted by γ_A.

group functor, we get a simplicial group model for $L(p,q)$:

$$\mathbb{G}[L(p,q)] \;\cong\; \mathrm{hocolim}\,[\,\mathbb{F}_1 \xleftarrow{\pi} \mathbb{G}(\mathbb{T}^2) \xrightarrow{\gamma} \mathbb{F}_1\,]\;. \tag{7.2.19}$$

Recall (see (7.1.2)) that $\mathbb{G}(\mathbb{T}^2)$ is given in $\mathtt{sGr}$ by the homotopy cofibre of the commutator map $\alpha : \mathbb{F}_1 \to \mathbb{F}_2,\ c \mapsto [a,b]$, where a and b are the generators of $\mathbb{F}_2$ corresponding to the meridian and longitude in $\mathbb{T}^2$. In terms of these generators, the maps π and γ in (7.2.19) are induced by

$$\pi:\ \mathbb{F}_2 \to \mathbb{F}_1\ ,\ (a,b) \mapsto (z,1)\ , \qquad \gamma:\ \mathbb{F}_2 \to \mathbb{F}_1\ ,\ (a,b) \mapsto (z^m, z^p)\ , \tag{7.2.20}$$

where z is a generator of $\mathbb{F}_1$.

Now, assume that G admits a global Koszul resolution $K_*(G)$ described in Section 7.1. Then, we have am explicit DG algebra model for $\mathrm{HR}_*(\mathbb{T}^2, G)$ given by $\mathcal{A}_*(\mathbb{T}^2, G) = \mathcal{O}(G \times G) \otimes_{\mathcal{O}(G)} K_*(G)$. Applying to (7.2.19) the derived representation functor, we get

$$\mathrm{DRep}_G[\mathbb{G}(L(p,q))] \;\cong\; \mathrm{hocolim}\,[\,\mathcal{O}(G) \xleftarrow{\pi_*} \mathcal{A}_*(\mathbb{T}^2, G) \xrightarrow{\gamma_*} \mathcal{O}(G)\,]\;, \tag{7.2.21}$$

The maps π_* and γ_* in (7.2.21) are determined by (7.2.20); on the degree 0 component of the DG algebra $\mathcal{A}_*(\mathbb{T}^2, G)$, they are given by

$$\pi_*:\ \mathcal{O}(G \times G) \to \mathcal{O}(G)\ , \quad f(x,y) \mapsto f(z,e)\ , \tag{7.2.22}$$

$$\gamma_*:\ \mathcal{O}(G \times G) \to \mathcal{O}(G)\ , \quad f(x,y) \mapsto f(z^m, z^p)\ . \tag{7.2.23}$$

Using π_* and γ_*, we can make $\mathcal{O}(G)$ into (left and right) DG modules over the DG algebra $\mathcal{A}_*(\mathbb{T}^2, G)$, which we denote by $\mathcal{O}(G)_\pi$ and $\mathcal{O}(G)_\gamma$ respectively. With this notation, we have the following result that completes our calculation.

Theorem 7.2.24. *The representation homology of a 3-dimensional lens space $L(p,q)$ is given by*

$$\mathrm{HR}_*(L(p,q), G) \;\cong\; \mathbf{Tor}_*^{\mathcal{A}_*}(\mathcal{O}(G)_\pi,\ \mathcal{O}(G)_\gamma)\ ,$$

where $\mathbf{Tor}_*^{\mathcal{A}_*}$ *denotes the differential* Tor *taken over the DG algebra* $\mathcal{A}_* = \mathcal{A}_*(\mathbb{T}^2, G)$. *In particular, there is an Eilenberg-Moore homology spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}_*^{\mathrm{HR}_*(\mathbb{T}^2,G)}(\mathcal{O}(G)_\pi,\, \mathcal{O}(G)_\gamma) \implies \mathrm{HR}_*(L(p,q),G)$$

converging to the representation homology of $L(p,q)$.

Closed 3**-manifolds**

The above construction of lens spaces generalizes to arbitrary closed 3-manifolds. Specifically, it is well known (see, e.g., [147]) that every closed connected orientable 3-manifold M admits a Heegaard decomposition $H_g \cup_\gamma H_g$ which can be written as

$$M \cong \operatorname{colim}[\, H_g \xleftarrow{i} \Sigma_g \xrightarrow{\gamma} H_g\,]\,, \tag{7.2.25}$$

where H_g is a handlebody of genus $g \geq 0$, i is the natural inclusion identifying $\Sigma_g = \partial H_g$ and γ is a gluing map defined as the composition $\Sigma_g \xrightarrow{\gamma_A} \Sigma_g \xhookrightarrow{i} H_g$, where γ_A is an (orientation-preserving) diffeomorphism of Σ_g representing an element in the mapping class group $\mathcal{M}(\Sigma_g) := \pi_0(\mathrm{Diff}^+\Sigma_g)$. In particular, for $g = 1$, the Heegaard diagram (7.2.25) becomes (7.2.17); in fact, the lens spaces can be characterized as (closed) 3-manifolds that admit Heegaard decompositions of genus 1.

Since H_g is homotopy equivalent as a cell complex to the bouquet of g circles $\vee_{i=1}^g \mathbb{S}^1$, we can represent the homotopy type of M by

$$M \cong \operatorname{hocolim}\left[\bigvee_{i=1}^{g}\mathbb{S}^1 \leftarrow \Sigma_g \rightarrow \bigvee_{i=1}^{g}\mathbb{S}^1\right].$$

This gives the simplicial group model $\mathbb{G}(M) \cong \operatorname{hocolim}[\,\mathbb{F}_g \xleftarrow{\pi} \mathbb{G}(\Sigma_g) \xrightarrow{\gamma} \mathbb{F}_g\,]$, and hence

$$\mathrm{DRep}_G[\mathbb{G}(M)] \;\cong\; \mathrm{hocolim}\,[\,\mathcal{O}(G^g) \xleftarrow{\pi_*} \mathcal{A}_*(\Sigma_g, G) \xrightarrow{\gamma_*} \mathcal{O}(G^g)\,] \;\cong\; \mathcal{O}(G^g) \otimes^{\boldsymbol{L}}_{\mathcal{A}_*} \mathcal{O}(G^g)\,,$$

where $\mathcal{A}_* = \mathcal{A}_*(\Sigma_g, G)$ is an explicit DG algebra model for the representation homology $\mathrm{HR}_*(\Sigma_g, G)$ (see Section 7.1). As a result, we have the following generalization of Theorem 7.2.24 to 3-manifolds of higher genus.

Theorem 7.2.26. *Let M a closed connected orientable 3-manifold. Assume that M has a Heegaard decomposition* (7.2.25) *determined by an element $\gamma \in \mathcal{M}(\Sigma_g)$ in the mapping class group of Σ_g. Then the representation homology of M is given by*

$$\mathrm{HR}_*(M, G) \;\cong\; \mathbf{Tor}^{\mathcal{A}_*}_*(\mathcal{O}(G^g)_\pi,\, \mathcal{O}(G^g)_\gamma)\;,$$

where $\mathbf{Tor}^{\mathcal{A}_*}_*$ is the differential Tor *taken over the DG algebra $\mathcal{A}_* = \mathcal{A}_*(\Sigma_g, G)$. In particular, there is an Eilenberg-Moore homology spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}^{\mathrm{HR}_*(\Sigma_g, G)}_*(\mathcal{O}(G^g)_\pi,\, \mathcal{O}(G^g)_\gamma) \implies \mathrm{HR}_*(M, G)$$

converging to the representation homology of M.

CHAPTER 8

REPRESENTATION COHOMOLOGY AND A NON-ABELIAN DENNIS TRACE MAP

In this section, we identify representation homology as the Hochschild-Mitchell (hyper)homology of a certain bifunctor on the category of finitely generated free groups $\mathfrak{G}$ with values in chain complexes of k-vector spaces.

8.1 Representation cohomology

(Co)homology of small categories

Let $\mathscr{C}$ be a small category. By a *$\mathscr{C}$-bimodule*, we mean a bifunctor $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathtt{Vect}_k$, which is contravariant in the first argument and covariant in the second. We write $\mathtt{Bimod}(\mathscr{C})$ for the category of $\mathscr{C}$-bimodules. For any $D \in \mathtt{Bimod}(\mathscr{C})$, one can define the (Hochschild-Mitchell) homology $\mathrm{HH}_*(\mathscr{C}, D)$ and cohomology $\mathrm{HH}^*(\mathscr{C}, D)$ of $\mathscr{C}$ with coefficients in D. For a precise definition and basic properties of these classical (co)homology theories we refer to [113, 30, 71, 72] (a good summary can also be found in [106, Appendix C]). Here, we only recall that $\mathrm{HH}_*(\mathscr{C}, -)$ and $\mathrm{HH}^*(\mathscr{C}, -)$ are functors (covariant and contravariant, respectively) on the category of $\mathscr{C}$-bimodules, such that $\{\mathrm{HH}_n(\mathscr{C}, -)\}_{n\geq 0}$ and $\{\mathrm{HH}^n(\mathscr{C}, -)\}_{n\geq 0}$ are universal δ-sequences, with $\mathrm{HH}_0(\mathscr{C}, D)$ and $\mathrm{HH}^0(\mathscr{C}, D)$ being canonically isomorphic to the coend $\int^{c\in\mathscr{C}} D(c, c)$ and the end $\int_{c\in\mathscr{C}} D(c, c)$ of the bifunctor D. Moreover, the (co)homology theories $\mathrm{HH}_*(\mathscr{C}, D)$ and $\mathrm{HH}^*(\mathscr{C}, D)$ have good functorial properties with respect to the first argument: in particular, any functor $F : \mathscr{C}' \to \mathscr{C}$ between small categories induces a natural map on homology $F_* : \mathrm{HH}_*(\mathscr{C}', F^*D) \to \mathrm{HH}_*(\mathscr{C}, D)$, where $F^* : \mathtt{Bimod}(\mathscr{C}) \to \mathtt{Bimod}(\mathscr{C}')$ is the restriction functor on bimodules defined by $F^*D := D \circ (F^{\mathrm{op}} \times F)$.

Representation cohomology

To express representation homology in terms of Hochschild-Mitchell homology, we need to slightly extend the above classical setting. Specifically, we will consider chain complexes of $\mathscr{C}$-bimodules, which are simply bifunctors $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathtt{Ch}_{\geq 0}(k)$ with values in the category of chain complexes of k-vector spaces, and define $\mathbb{HH}_*(\mathscr{C}, D)$ and $\mathbb{HH}^*(\mathscr{C}, D)$ to be the Hochschild-Mitchell *hyperhomology* and the Hochschild-Mitchell *hypercohomology* of D, respectively. Now, given two chain complexes of right and left $\mathscr{C}$-modules, say $M : \mathscr{C}^{\mathrm{op}} \to \mathtt{Ch}_{\geq 0}(k)$ and $N : \mathscr{C} \to \mathtt{Ch}_{\geq 0}(k)$, we define the chain complex of $\mathscr{C}$-bimodules $M \boxtimes_k N : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathtt{Ch}_{\geq 0}(k)$ by assigning to $(c, c') \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C})$ the tensor product $M(c) \otimes_k N(c')$ of the corresponding chain complexes. With this notation, we have

Lemma 8.1.1. *For any $X \in \mathtt{sSet}_0$ and any commutative Hopf algebra $\mathcal{H}$, there is a natural isomorphism*

$$\mathrm{HR}_*(X, \mathcal{H}) \cong \mathbb{HH}_*(\mathfrak{G},\, \underline{N}(k[\mathbb{G}X]) \boxtimes_k \underline{\mathcal{H}}\,)\,. \tag{8.1.2}$$

Proof. For any small category $\mathscr{C}$ and any $\mathscr{C}$-modules M and N with values in $\mathtt{Ch}_{\geq 0}(k)$, where k is a commutative ring, there is a natural (Grothendieck) spectral sequence (see, e.g., [106, (C.10.1)]):

$$E^2_{pq} = \mathbb{HH}_p(\mathscr{C},\, \mathrm{H}_q[M \boxtimes^{\boldsymbol{L}}_k N]) \implies \mathrm{H}_{p+q}[M \otimes^{\boldsymbol{L}}_{\mathscr{C}} N]\,.$$

When k is a field, this spectral sequence degenerates giving an isomorphism $\mathbb{HH}_*(\mathscr{C},\, M \boxtimes_k N) \cong \mathrm{H}_*[M \otimes^{\boldsymbol{L}}_{\mathscr{C}} N]$. In our situation, we have

$$\mathbb{HH}_*(\mathfrak{G},\, \underline{N}(k[\mathbb{G}X]) \boxtimes_k \underline{\mathcal{H}}\,) \;\cong\; \mathrm{H}_*[\underline{N}(k[\mathbb{G}X]) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \underline{\mathcal{H}}]\ ,$$

which in composition with the isomorphism of Theorem 3.3.3 gives (8.1.2). $\square$

Lemma 8.1.1 motivates the following definition.

Definition 8.1.3. The *representation cohomology* of X in $\mathcal{H}$ is defined by

$$\mathrm{HR}^*(X, \mathcal{H}) \;:=\; \mathbb{HH}^*(\mathfrak{G},\, \underline{N}(k[\mathbb{G}X]) \boxtimes_k \underline{\mathcal{H}}\,)$$

More generally, for any $\mathfrak{G}$-bimodule $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \;\to\; \mathtt{Ch}_{\geq 0}(k)$, we define the *representation homology* and the *representation cohomology* of D by

$$\mathrm{HR}_*(D) := \mathbb{HH}_*(\mathfrak{G},\, D)\ , \quad \mathrm{HR}^*(D) := \mathbb{HH}^*(\mathfrak{G},\, D)\ .$$

In the case when D is an ordinary $\mathfrak{G}$-bimodule (with values in $\mathtt{Vect}_k$), this definition says that the representation (co)homology of D is just the classical Hochschild-Mitchell (co)homology of D.

Relation to topological Hochschild homology

For an arbtirary (associative unital) ring R, denote by $F(R)$ the full subcategory of R-$\mathtt{Mod}$ whose objects are the free modules $R^n,\ n \geq 0$. For any R-bimodule N, consider the bifunctor $\mathrm{Hom}(I, N)\,:\, F(R)^{\mathrm{op}} \times F(R) \to \mathtt{Mod}(\mathbb{Z})$ defined by $(X, Y) \mapsto \mathrm{Hom}_R(X, N \otimes_R Y))$. Then, a theorem of Pirashvili and Waldhausen [132] asserts that the Hochschild-Mitchell homology $\mathrm{HH}_*(F(R), \mathrm{Hom}(I, N))$ is naturally isomorphic to the *topological Hochschild homology* $\mathrm{THH}_*(R, N)$ of the ring R with coefficients in the bimodule N. It is therefore natural to *define* the topological Hochschild homology of R with coefficients in an arbitrary bifunctor $B:\ F(R)^{\mathrm{op}} \times F(R) \to \mathtt{Mod}(\mathbb{Z})$ by (*cf.* [106, Chap. 13])

$$\mathrm{THH}_*(R, B) := \mathrm{HH}_*(F(R),\ B)\ .$$

For $R = \mathbb{Z}$, the category $F(\mathbb{Z})$ is equivalent to the category $\mathfrak{G}_{\mathrm{ab}}$ of finitely generated free abelian groups, which (as our notation suggests) is the abelianization of the category $\mathfrak{G}$. The abelianization functor $\alpha : \mathfrak{G} \to \mathfrak{G}_{\mathrm{ab}}$ induces a natural map $\mathrm{HR}_*(\alpha^* B) \to \mathrm{THH}_*(\mathbb{Z}, B)$ for any $\mathfrak{G}_{\mathrm{ab}}$-bimodule $B \in \mathtt{Bimod}(\mathfrak{G}_{\mathrm{ab}})$, and conversely, for any $\mathfrak{G}$-bimodule $D \in \mathtt{Bimod}(\mathfrak{G})$, associated to the functor α, there is an André-type spectral sequence (see [71, Theorem 1.20]):

$$E^2_{pq} = \mathrm{THH}_p(\mathbb{Z},\ \boldsymbol{L}_q(\alpha^{\mathrm{op}} \times \alpha)_* D) \implies \mathrm{HR}_{p+q}(D)\ ,$$

converging to the representation homology of D.

Thus, representation homology may be viewed as a non-abelian analogue of topological Hochschild homology, and it is natural to ask for 'non-abelian' analogues of various constructions known for topological Hochschild homology. In the next section, we outline one such construction which may be thought of as a non-abelian version of the Dennis trace map.

8.2 Non-abelian Dennis trace map

Recall[1] that the classical Dennis trace maps the stable homology of the general linear groups of a ring R to topological Hochschild homology of R:

$$\mathrm{DTr}_\infty(R, B):\ \mathrm{H}_*(\mathrm{GL}_\infty(R),\ B_\infty)\ \to\ \mathrm{THH}_*(R, B)\ , \tag{8.2.1}$$

[1]See [106, Sect. 13.1.8] for the case $B = \mathrm{Hom}(I, N)$ and [61] for an extension to arbitrary $F(R)$-bimodules.

where B is an arbitrary bimodule over $F(R)$. We generalize this map to the non-abelian setting.

Let $\mathrm{Aut}_n := \mathrm{Aut}(\mathbb{F}_n)$ denote the automorphism group of the free group on generators $x_1, \ldots, x_n$. We will regard Aut_n as the automorphism group $\mathrm{Aut}_{\mathfrak{G}}(\langle n \rangle)$ of the object $\langle n \rangle$ in the category $\mathfrak{G}$. There are obvious inclusions $\mathrm{Aut}_n \hookrightarrow \mathrm{Aut}_{n+1}$ defined by $g \mapsto \tilde{g}$, where $\tilde{g}(x_i) := g(x_i)$ for $i \leq n$ and $\tilde{g}(x_{n+1}) = x_{n+1}$. We set $\mathrm{Aut}_\infty := \varinjlim \mathrm{Aut}_n$.

Now, consider an arbitrary bimodule D on the category $\mathfrak{G}$, i.e. a bifunctor $D : \mathfrak{G}^{\mathrm{op}} \times \mathfrak{G} \to \mathtt{Vect}_k$. For each $n \geq 1$, let $D_n := D(\langle n \rangle, \langle n \rangle)$ and define the linear maps

$$p_n^* \circ (i_n)_* : \ D_n \to D(\langle n \rangle, \langle n+1 \rangle) \to D_{n+1} \ , \tag{8.2.2}$$

where $(i_n)_* := D(\mathrm{id}, i_n)$ and $p_n^* := D(p_n, \mathrm{id})$ are induced by the natural inclusion $i_n : \langle n \rangle \hookrightarrow \langle n+1 \rangle$ and the natural projection $p_n : \langle n+1 \rangle \twoheadrightarrow \langle n \rangle$, respectively. Put

$$D_\infty := \varinjlim D_n \ ,$$

where the inductive limit is taken with respect to (8.2.2).

Next, observe that each D_n carries a natural Aut_n-module structure: namely, $\mathrm{Aut}_n \to \mathrm{Aut}_k D(\langle n \rangle, \langle n \rangle))$, $g \mapsto g^* \circ g_*$, where $g^* := D(g, \mathrm{id})$ and $g_* := D(\mathrm{id}, g)$. Moreover, for all $g \in \mathrm{Aut}_n$, there is a commutative diagram

$$\begin{tikzcd} D_n \arrow[r, "g^*\, g_*"] \arrow[d, "p_n^*\,(i_n)_*"'] & D_n \arrow[d, "p_n^*\,(i_n)_*"] \\ D_{n+1} \arrow[r, "\tilde{g}^*\, \tilde{g}_*"] & D_{n+1} \end{tikzcd}$$

As a consequence, the k-vector space D_∞ carries a natural (inductive) Aut_∞-module structure. Thus, we can consider the homology groups $\mathrm{H}_*(\mathrm{Aut}_n, D_n)$

for all $n \geq 1$ and $\mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty)$. Since homology commutes with direct limits, we can identify

$$\mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty) \;\cong\; \varinjlim \mathrm{H}_*(\mathrm{Aut}_n, D_n)\ . \tag{8.2.3}$$

Next, we construct natural maps relating $\mathrm{H}_*(\mathrm{Aut}_n, D_n)$ and $\mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty)$ to representation homology $\mathrm{HR}_*(D)$. We proceed as in [106, 13.1.8], but with one important modification. Instead of using the Mac Lane Isomorphism, which identifies group homology with classical Hochschild homology in the case of GL_n and which is not available in our non-abelian setting, we will use the Baues-Wirsching Isomorphism, which identifies group homology with Baues-Wirsching homology [71]. Specifically, regarding each group Aut_n as a category $\mathfrak{Aut}_n$ with a single object, we consider the natural functors

$$\gamma_n :\ \mathfrak{Aut}_n \;\to\; \mathfrak{G}\ ,$$

mapping the single object $* \in \mathfrak{Aut}_n$ to $\langle n\rangle \in \mathfrak{G}$. Then, by [71, Remark 1.10], we have natural isomorphisms

$$\mathrm{H}_*(\mathrm{Aut}_n, D_n) \;\cong\; \mathrm{H}_*^{\mathrm{BW}}(\mathfrak{Aut}_n,\, \gamma_n^* D) \;\cong\; \mathrm{Tor}_*^{\mathcal{F}\mathfrak{Aut}_n^{\mathrm{op}}}(\mathbb{Z}, \gamma_n^* D)\ ,$$

where $\mathcal{F}\mathscr{C}$ denotes the Baues-Wirsching factorization category of a small category $\mathscr{C}$ (see [30]). Similarly, by [71, Proposition 1.13], there are natural isomorphisms

$$\mathrm{HR}_*(D) := \mathrm{HH}_*(\mathfrak{G}, D) \;\cong\; \mathrm{H}_*^{\mathrm{BW}}(\mathfrak{G}, D) \;\cong\; \mathrm{Tor}_*^{\mathcal{F}\mathfrak{G}^{\mathrm{op}}}(\mathbb{Z}, D)\ .$$

Now, observe that, for each $n \geq 0$ and any $\mathfrak{G}$-bimodule D, there is a canonical map $\mathrm{H}_0(\mathrm{Aut}_n, D_n) \to \mathrm{HR}_0(D) \cong \int^{n\geq 0} D_n$ induced by embedding each D_n into the coend diagram of the bifunctor D. Since γ_n^* is an exact functor from the category of left $\mathcal{F}\mathfrak{G}^{\mathrm{op}}$-modules to the category of left $\mathfrak{Aut}_n$-modules, the sequence $\{\mathrm{Tor}_i^{\mathcal{F}\mathfrak{Aut}_n^{\mathrm{op}}}(\mathbb{Z}, \gamma_n^*(-))\}_{i\geq 0}$ forms a covariant left δ-functor. Since

$\{\mathrm{Tor}_i^{\mathcal{F}\,\mathfrak{Aut}_n^{\mathrm{op}}}(\mathbb{Z},-)\}_{i\geq 0}$ is a universal δ-functor, the map $\mathrm{H}_0(\mathrm{Aut}_n, D_n) \to \mathrm{HR}_0(D)$ extends to canonical linear maps

$$\mathrm{DTr}_n^{\mathfrak{G}}(D):\ \mathrm{H}_i(\mathrm{Aut}_n, D_n)\ \to\ \mathrm{HR}_i(D)\ , \quad \forall\, i \geq 0\ . \tag{8.2.4}$$

As in [106, 13.1.8], it is easy to check that the maps (8.2.4) are compatible when passing from n to $n+1$. Hence, we can stabilize (8.2.4) by passing to the inductive limit as $n \to \infty$. With identification (8.2.3), the resulting stable map reads

$$\mathrm{DTr}_\infty^{\mathfrak{G}}(D):\ \mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty)\ \to\ \mathrm{HR}_*(D)\ . \tag{8.2.5}$$

This is a non-abelian analogue of the classical Dennis trace map (8.2.1). As in the classical case, it is natural to ask: When is (8.2.5) an isomorphism? Motivated by a theorem of Scorichenko [146] (see also [61]), we propose a conjectural answer:

Conjecture 8.2.6. *The map* (8.2.5) *is an isomorphism if D is a polynomial*[2] *bifunctor.*

We conclude this section a few remarks related to Conjecture 8.2.6.

1. A famous theorem of Galatius [73] asserts that natural maps from the symmetric group S_n to Aut_n (defined by permuting the generators) induce isomorphisms

$$\mathrm{H}_i(\mathrm{Aut}_n,\,\mathbb{Z})\ \cong\ \mathrm{H}_i(S_n,\,\mathbb{Z})\ , \quad \forall\, n > 2i+1\ .$$

This implies that $\mathrm{H}_i(\mathrm{Aut}_\infty,\,A) = 0$ for all $i > 0$, where A is any constant k-module provided k has characteristic 0 (which we always assume in Part I). Our Conjecture 8.2.6 implies [50, Theorem 1], which states that $\mathrm{H}_i(\mathrm{Aut}_\infty,\,D_\infty) = 0$

[2]Strictly speaking, the notion of a polynomial functor is usually defined in the literature for functors $T : \mathscr{C} \to \mathscr{A}$ between two additive categories (see, e.g., [106, E.13.1.3]). However, the definition of the n-th cross-effect, in terms of which one defines polynomial degrees, makes sense for functors T whose domain $\mathscr{C}$ is any pointed cocartesian monoidal category; in particular, it applies to the category $\mathfrak{G}$ (see [82]).

for all $i > 0$, when D is a polynomial bifunctor, which is *constant* with respect to the contravariant argument. Indeed, for such bifunctors, we have $\mathrm{HR}_i(D) = \mathrm{HH}_i(\mathfrak{G}, D) = 0$ for $i > 0$ because $\mathfrak{G}$ has a terminal object.

2. The direct analogue of Conjecture 8.2.6 is false in the abelian case. (Indeed, if B is a constant bifunctor on $F(R)$, then $\mathrm{THH}_*(R, B)$ vanishes in positive degrees (since $F(R)$ has terminal object), but $\mathrm{H}_*(\mathrm{GL}_\infty(R),\, B)$ may be highly nontrivial, see [61].) The correct version of Conjecture 8.2.6 proved by Scorichenko [146] replaces the stable group homology with Waldhausen's stable K-theory. In the non-abelian case, one case also state a version of Conjecture 8.2.6 for the stable K-theory of automorphism groups Aut_n instead of group homology; however, we expect that in this case, the two theories are actually isomorphic. We briefly outline an argument behind this expectation.

Let E_∞ denote the commutator subgroup of Aut_∞. It is known that E_∞ is a perfect normal subgroup, hence we can form the 'plus construction'

$$\Psi :\; B\mathrm{Aut}_\infty \;\to\; B\mathrm{Aut}_\infty^+ \,.$$

Let $F\Psi$ denote the homotopy fiber of the map Ψ. We have a canonical group homomorphism $\pi_1(F\Psi) \to \pi_1(B\mathrm{Aut}_\infty) \cong \mathrm{Aut}_\infty$ that equips any Aut_∞-module with a $\pi_1(F\Psi)$-action. In particular, the Aut_∞-module D_∞ arising from a $\mathfrak{G}$-bimodule D may be viewed as a $\pi_1(F\Psi)$-module, and hence defines a local system on $F\Psi$. The *stable K-theory* $K^s_*(\mathrm{Aut}_\infty, D_\infty)$ is then defined to be $\mathrm{H}_*(F\Psi,\, D_\infty)$, the homology of $F\Psi$ with coefficients in the local system D_∞. Now, consider the Serre spectral sequence associated to the homotopy fibration $F\Psi \to B\mathrm{Aut}_\infty \to B\mathrm{Aut}_\infty^+$:

$$E^2_{pq} \;=\; \mathrm{H}_p(B\mathrm{Aut}_\infty^+,\, \mathrm{H}_q(F\Psi, D_\infty)) \;\Longrightarrow\; \mathrm{H}_n(B\mathrm{Aut}_\infty,\, D_\infty) \,.$$

If Aut_∞ acts *trivially* on $K^s_q(\mathrm{Aut}_\infty,\, D_\infty) \;=\; \mathrm{H}_q(F\Psi,\, D_\infty)$ (as it happens in the classical case, see [106, 13.3.2]), then, since $B\mathrm{Aut}_\infty \;\to\; B\mathrm{Aut}^+_\infty$ is a homology equivalence for trivial coefficients, the above spectral sequence becomes

$$E^2_{pq} \;=\; \mathrm{H}_p(B\mathrm{Aut}_\infty,\, K^s_q(\mathrm{Aut}_\infty,\, D_\infty)) \;\Longrightarrow\; \mathrm{H}_n(B\mathrm{Aut}_\infty,\, D_\infty)\,.$$

However, by Galatius' Theorem [73], we know that $\mathrm{H}_p(\mathrm{Aut}_\infty, A) \;=\; 0$ for $p > 0$ for any constant coefficients over k. Hence, the above spectral sequence must collapse on the p-axis, giving a desired isomorphism $K^s_*(\mathrm{Aut}_\infty, D_\infty) \;\cong\; \mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty)$.

Part II

Derived representation schemes of simplicial groups

CHAPTER 9

SIMPLICIAL ALGEBRAS

9.1 Simplicial objects in algebraic categories

A group can be defined to be a set G with three operations: a binary operation $\cdot : G \times G \to G$, a 0-ary operation $e \in G$, as well as a 1-ary operation $(-)^{-1} : G \to G$, satisfying the identities

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$x \cdot e = x = e \cdot x$$

$$x \cdot x^{-1} = e = x^{-1} \cdot x$$

More generally, any specified set of finitary operations and identities defines an *algebraic theory* (see [14, Chapter 3] and [104] for a precise formualtion), which in turn allows one to define algebras over such an algebraic theory. A category equivalent to the category of such algebras is said to be an *algebraic category*. Any algebraic category $\mathcal{C}$ comes equipped with a forgetful functor $U : \mathcal{C} \to \mathtt{Set}$, which moreover always have a left adjoint $F : \mathtt{Set} \to \mathcal{C}$.

Given an algebraic category $\mathcal{C}$, we will consider the category $\mathtt{s}\mathcal{C}$ of simplicial objects in $\mathcal{C}$. The first thing to notice about this category is that it has a natural simplicial structure. By a *simplicial structure* on a category $\mathcal{M}$, we mean an enrichment $\tilde{\mathcal{M}}$ of $\mathcal{M}$ over simplicial sets, such that $\tilde{\mathcal{M}}$ is powered and copowered, and that $\mathcal{M} = (\tilde{\mathcal{M}})_0$. More precisely, this means that there is a simplicial set $\underline{\mathrm{Hom}}_{\tilde{\mathcal{M}}}(X, Y)$ for each pair of objects $X, Y \in \mathcal{M}$, together with an associative composition law between them, such that, for each $K \in \mathtt{sSet}$, there is a powering functor $(-)^K : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ and a copowering functor $K \otimes - : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ such that the adjunction

$$\underline{\mathrm{Hom}}_{\tilde{\mathcal{M}}}(K \otimes X, Y) \cong \underline{\mathrm{Hom}}_{\mathtt{sSet}}(K, \underline{\mathrm{Hom}}_{\tilde{\mathcal{M}}}(X, Y)) \cong \underline{\mathrm{Hom}}_{\tilde{\mathcal{M}}}(X, Y^K) \tag{9.1.1}$$

holds naturally in $X, Y \in \mathcal{M}$. Clearly, the three structures: the enrichment, the copowering, and the powering determine each other by the adjunction (9.1.1). For example, given only the (unenriched) copowering operations $K \otimes - : \mathcal{M} \to \mathcal{M}$, one can determine the simplicial enrichment by noticing that the n-th simplicial degree of the Hom space $\underline{\mathrm{Hom}}_{\tilde{\mathcal{M}}}(X, Y)$ is simply given by

$$\mathrm{Hom}_{\mathtt{sSet}}(\Delta^n, \underline{\mathrm{Hom}}_{\tilde{\mathcal{M}}}(X, Y)) \cong \mathrm{Hom}_{\mathcal{M}}(\Delta^n \otimes X, Y)$$

As was briefly alluded to above, the category $\mathtt{s}\mathcal{C}$ of simplicial objects in an algebraic category $\mathcal{C}$ admits a natural simplicial structure. For each $A \in \mathtt{s}\mathcal{C}$ and each $K \in \mathtt{sSet}$, define $K \otimes A \in \mathtt{s}\mathcal{C}$ to be the simplicial object given in simplicial degree n by the coproduct

$$(K \otimes A)_n := \coprod_{K_n} A_n$$

of copies of A_n indexed by the set K_n. The face and degeneration maps of $K \otimes A$ will then be determined in the obvious way by those of K and A. This gives a bifunctor $\mathtt{sSet} \times \mathtt{s}\mathcal{C} \to \mathtt{s}\mathcal{C}$ which, as discussed above, determines a simplicial structure, called the *standard simplicial structure*, on $\mathtt{s}\mathcal{C}$.

It is well known (see, e.g., Remark 1 after Proposition 1 of [135, Section II.4]) that if $\mathcal{C}$ is an algebraic category, then the category $\mathtt{s}\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has a model structure induced from the model structure on the category $\mathtt{sSet}$ of simplicial sets. More precisely, we have the following

Theorem 9.1.2. *Let $\mathcal{C}$ be an algebraic category, with the forgetful functor $U : \mathcal{C} \to$*

$\mathtt{Set}$ *to the category of sets, then the category* $\mathtt{s}\mathcal{C}$ *of simplicial objects in* $\mathcal{C}$ *has a model structure where*

1. *A map* $f : X \to Y$ *in* $\mathtt{s}\mathcal{C}$ *is a fibration if and only if its underlying map* $U(f) : U(X) \to U(Y)$ *of simplicial sets is a fibration.*
2. *A map* $f : X \to Y$ *in* $\mathtt{s}\mathcal{C}$ *is a weak equivalence if and only if its underlying map* $U(f) : U(X) \to U(Y)$ *of simplicial sets is a weak equivalence.*
3. *A map* $f : X \to Y$ *in* $\mathtt{s}\mathcal{C}$ *is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations.*

It is then immediately clear that the adjunction

$$F : \mathtt{sSet} \rightleftarrows \mathtt{s}\mathcal{C} : U \tag{9.1.3}$$

is a Quillen adjunction.

The simplicial structure and the model structure on $\mathtt{s}\mathcal{C}$ are compatible, making it into a simplicial model category, in the following sense:

Definition 9.1.4. A *simplicial model category* is a model category $\mathcal{M}$ with a simplicial structure such that the copowering bifunctor $\otimes : \mathtt{sSet} \times \mathcal{M} \to \mathcal{M}$ is a left Quillen bifunctor. This means that, for each cofibration $i : A \hookrightarrow B$ in $\mathcal{M}$ and each cofibration $j : K \hookrightarrow L$ in $\mathtt{sSet}$, their smash product

$$i \wedge j : K \otimes B \amalg_{K \otimes A} L \otimes A \to L \otimes B$$

is a cofibration, and is moreover a trivial cofibration if either i or j is.

We remark that, since a simplicial structure is determiend by either the simplicial enrichment, the powering or the copowering, one can alternatively express the above compatibility condition between the model structure and the simplicial structure in terms of either the powering or the simplicial enrichment (see, e.g., [79, Lemma 4.2.2]).

In Theorem 9.1.2, we have identified the class of cofibrations in an indirect way, via their left lifting property with respect to trivial fibrations. One can in fact describe the cofibrations as retracts of semi-free extensions, to be defined below. Besides giving ample supply of cofibrant objects and cofibrations, the classes of semi-free objects and semi-free extensions admit simple descriptions that often capture the essential information for the situation in hand.

Definition 9.1.5. Given an algebraic category $\mathcal{C}$ with forgetful functor $U : \mathcal{C} \to \mathtt{Set}$ and its left adjoint $F : \mathtt{Set} \to \mathcal{C}$, a map $f : X \to Y$ in $\mathrm{s}\mathcal{C}$ is said to be a *semi-free extension* if there is a collection $B_n \subset U(Y_n)$ of subsets of the underlying sets $U(Y_n)$ of Y_n, such that the union $\{B_n\}_{n\geq 0}$ is closed under degeneration maps of the simplicial set $U(Y_n)$, and such that each Y_n is obtained from X_n by freely adjoining the variables B_n, *i.e.*, the canonical map $X_n \amalg F(B_n) \to Y_n$ in $\mathcal{C}$ is an isomorphism.

An object $Y \in \mathrm{s}\mathcal{C}$ is said to be *semi-free* if the canonical map $F(\emptyset) \to Y$ from the initial object $F(\emptyset)$ to Y is a semi-free extension.

Proposition 9.1.6. *A map $f : X \to Y$ in $\mathrm{s}\mathcal{C}$ is a cofibration if and only if it is a retract of a semi-free extension. In particular, an object $Y \in \mathrm{s}\mathcal{C}$ is cofibrant if and only if it is a retract of a semi-free object.*

To prove this statement, one begins with the following alternative characterization of semi-free extensions:

Proposition 9.1.7. *A map $f : X \to Y$ in $\mathrm{s}\mathcal{C}$ is a semi-free extension if and only if it* *can be written as an infinite composition*

$$X = \mathrm{sk}_{-1}(f) \to \mathrm{sk}_0(f) \to \mathrm{sk}_1(f) \to \ldots \to \varinjlim \mathrm{sk}_*(f) = Y$$

where each map $\mathrm{sk}_{n-1}(f) \to \mathrm{sk}_n(f)$ *is a pushout*

$$\begin{tikzcd} F(\partial\Delta^n) \arrow[r] \arrow[d] & F(\Delta^n) \arrow[d] \\ \mathrm{sk}_{n-1}(f) \arrow[r] & \mathrm{sk}_n(f) \end{tikzcd} \tag{9.1.8}$$

With this characterization of semi-free extensions, Proposition 9.1.6 follows from general results in cofibrantly generated model categories, which we review now.

Given a class I of morphisms in a category $\mathcal{M}$, we denote the class of morphisms satisfying left [resp. right] lifting property with morphisms in I by ${}^{\pitchfork}I$ [resp. $I^{\pitchfork}$]. For example, if I is the class of cofibrations in a model category, then $I^{\pitchfork}$ is the class of trivial fibrations, while ${}^{\pitchfork}(I^{\pitchfork})$ recovers the class of cofibration. Very often, one could start with a smaller class – in fact, a set – of morphisms I such that ${}^{\pitchfork}(I^{\pitchfork})$ is the class of cofibration. In this case, we say that the class of cofibrations is *generated by* the set I.

Notice that, given a class of morphisms P, the class ${}^{\pitchfork}P$ is always *weakly saturated*, which means that it is closed under coproducts, pushouts, retracts and transfinite composition [77, Definition 10.2.2]. Thus, given a set I of morphisms, the class ${}^{\pitchfork}(I^{\pitchfork})$ always contains the weak saturation of I. We say that a map $f : X \to Y$ is a *relative I-cell complex* if it can be represented as a transfinite composition of pushouts of coproducts of maps in I. Then the weak saturation of I can be described as the maps that are retracts of relative I-cell complexes. In many cases, the class ${}^{\pitchfork}(I^{\pitchfork})$ in fact equals the weak saturation of I. This holds, for example, when the set I *admits small object argument*, which means that the domains of the morphisms in I are α-small with respect to I for some regular cardinal α (see [77, Definition 10.4.1, 10.5.15, Corollary 10.5.23]).

A model category $\mathcal{M}$ is said to be *cofibrantly generated* [77, Definition 11.1.2] if there are sets I and J of morphisms that admits small object argument, such that $I^{\pitchfork}$ is the class of trivial fibrations (so that I generates the class of cofibrations), while $J^{\pitchfork}$ is the class of fibrations (so that J generates the class of trivial cofibrations). The model category sSet of simplicial set constitutes a basic and important example of cofibrantly generated model category. For example, one can take the set[1] $I = \{\, \partial\Delta^n \hookrightarrow \Delta^n \,\}_{n\geq 0}$ as the set of generating cofibrations, and the set $J = \{\, \Lambda^n_i \hookrightarrow \Delta^n \,\}_{n\geq 1}$ of horn inclusions as the set of generating trivial cofibrations. More generally, if $\mathcal{C}$ is an algebraic category, then the model category s$\mathcal{C}$ of simplicial objects in $\mathcal{C}$ is also cofibrantly generated, with the set $F(I)$ as generating cofibrations, and the set $F(J)$ as generating trivial cofibrations. In this case, a relative I-cell inclusion is precisely a transfinite composition of pushouts of the form (9.1.8). Such a transfinite composition can always be rearranged according to the simplicial degree of the "cells" attached. In other words, a relative I-cell inclusion in s$\mathcal{C}$ is precisely the infinite compositions that are identified as semi-free extensions in Proposition 9.1.7. Hence, Proposition 9.1.6 follows from our above discussion.

[1]Here, $\partial\Delta^0$ is the empty simplicial set.

9.2 Adapted objects

If $f : A \to B$ is a map of associative algebra, then to compute the image of a chain complex $M \in \mathrm{Ch}(A)$ under the derived functor $- \otimes^{\boldsymbol{L}} B : \mathcal{D}(A) \to \mathcal{D}(B)$, it suffices to resolve M by degreewise flat chain complexes, instead of projective complexes. In this case, we say that a chain complex of flat modules is adapted under the functor $- \otimes B : \mathrm{Ch}(A) \to \mathrm{Ch}(B)$.

More generally, suppose that $\mathcal{M}$ and $\mathcal{N}$ are model categories, and $F : \mathcal{M} \to \mathcal{N}$ is a functor that maps weak equivalences between cofibrant objects to weak equivalences, then F has a total left derived functor $\boldsymbol{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$, so that for each $X \in \mathcal{M}$, the image $\boldsymbol{L}F(\gamma(X)) \in \mathrm{Ho}(\mathcal{N})$ is calculated as $\boldsymbol{L}F(\gamma(X)) \cong \gamma(F(Q))$, where Q is a cofibrant replacement of X, and γ is the localization functor from a model category to its homotopy category (see, e.g., [77, Proposition 8.4.4]). In this situation, we make the following

Definition 9.2.1. An object $X \in \mathcal{M}$ is said to be *adapted* under F (or F-adapted), if the canonical map $\boldsymbol{L}F(\gamma(X)) \to \gamma(F(X))$ is an isomorphism in the homotopy category $\mathrm{Ho}(\mathcal{N})$.

While every cofibrant object in $\mathcal{M}$ is F-adapted, the converse is not true in general. This is illustrated by the example at the beginning of this subsection. The main goal of this subsection is to obtain a non-abelian analogue of this example (see Theorem 9.2.2 below).

To this end, let $\mathcal{C}$ and $\mathcal{D}$ be algebraic categories, so that the categories $\mathrm{s}\mathcal{C}$ and $\mathrm{s}\mathcal{D}$ of simplicial objects in them can be endowed with the model structures described in Theorem 9.1.2. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint functor, whose simplicial prolongation $F : \mathrm{s}\mathcal{C} \to \mathrm{s}\mathcal{D}$ maps weak equivalence between cofibrant objects to weak equivalences, so that F has a total left derived functor $\boldsymbol{L}F : \mathrm{Ho}(\mathtt{s}\mathcal{C}) \to \mathrm{Ho}(\mathtt{s}\mathcal{D})$. Then, we have

Theorem 9.2.2. *Suppose that $X \in \mathtt{s}\mathcal{C}$ is degreewise F-adapted, i.e., each $X_n \in \mathcal{C}$ is F-adapted when considered as a discrete simplicial object in $\mathtt{s}\mathcal{C}$, then X is F-adapted.*

Remark 9.2.3.
1. In this theorem, we do not assume that the functor $F : \mathtt{s}\mathcal{C} \to \mathtt{s}\mathcal{D}$ is a left Quillen functor. Thus, it is applicable to the representation functor $(-)_G : \mathtt{sGr} \to \mathtt{sCommAlg}_k$, to be defined below.
2. In the example at the beginning of this subsection, if one is willing to restrict to non-negatively graded chain complexes, then this theorem gives an honest generalization of that example.

To prove this theorem, we will need to consider bisimplicial objects in an algebraic category $\mathcal{C}$. A *bisimplicial object* in $\mathcal{C}$ is defined to be a simplicial object in the category $\mathtt{s}\mathcal{C}$, the category of such is denoted by $\mathtt{ss}\mathcal{C}$. Thus, a bisimplicial object is a functor $\Delta^{\mathrm{op}} \to \mathtt{s}\mathcal{C}$, given, say, by $[p] \mapsto X_{p,*} \in \mathtt{s}\mathcal{C}$. In other words, a bisimplicial object consists of a system of objects $X_{p,q} \in \mathcal{C}$, with simplicial face and degeneracy maps acting on both the *external degree* p and the *internal degree* q.

Given a bisimplicial object $X \in \mathtt{ss}\mathcal{C}$, one can focus on its diagonal part $\{X_{n,n}\}_{n\in\mathbb{N}}$, which clearly forms a simplicial object in $\mathcal{C}$. Thus, this gives the *diagonal functor*

$$\mathrm{diag} : \mathtt{ss}\mathcal{C} \to \mathtt{s}\mathcal{C}, \qquad \{X_{p,q}\}_{p,q\in\mathbb{N}} \mapsto \{X_{n,n}\}_{n\in\mathbb{N}} \tag{9.2.4}$$

The following is a basic result about this diagonal functor:

Proposition 9.2.5. *Let $f : X \to Y$ be a map of bisimplicial objects in $\mathcal{C}$. Suppose that either*

1. *For each fixed external degree p, the map $f : X_{p,*} \to Y_{p,*}$ is a weak equivalence in* $\mathtt{s}\mathcal{C}$*; or*

2. *For each fixed internal degree q, the map $f : X_{*,q} \to Y_{*,q}$ is a weak equivalence in* $\mathtt{s}\mathcal{C}$*,*

then the induced map $\operatorname{diag}(f) : \operatorname{diag}(X) \to \operatorname{diag}(Y)$ *is a weak equivalence in* $\mathtt{s}\mathcal{C}$*.*

Proof. A morphism in $\mathtt{s}\mathcal{C}$ is a weak equivalence if and only if its underlying map of simplicial sets is a weak equivalence. Therefore, one suffices to prove the theorem for the case $\mathcal{C} = \mathtt{Set}$. By the symmetry between the internal and external degrees, it suffices to prove the theorem under the assumption (1), which follows from [77, Theorem 15.11.6, 15.11.11] $\square$

The category $\mathtt{ss}\mathcal{C}$ is defined as the functor category $\mathtt{ss}\mathcal{C} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathtt{s}\mathcal{C})$. Since $\mathtt{s}\mathcal{C}$ is a cofibrantly generated model category, one can endow the functor category $\mathtt{ss}\mathcal{C}$ with a model structure, called *projective model structure*, such that weak equivalences in $\mathtt{ss}\mathcal{C}$ are maps $f : X \to Y$ that induces weak equivalence $X_{p,*} \to Y_{p,*}$ for each fixed simplicial degree. We now describe this projective model structure on functor categories.

Thus, let $\mathcal{M}$ be a cofibrantly generated model category, with a set I of generating cofibrations and a set J of generating fibrations, and let $\mathcal{D}$ be a small category. A map (*i.e.*, natural transformation) $F \to G$ between functors $F, G \in \mathcal{M}^{\mathcal{D}}$ is said to be a *objectwise weak equivalence* [resp. *objectwsie fibration*] if the induced map $F(x) \to G(x)$ in $\mathcal{M}$ is a weak equivalence [resp. fibration] for each $x \in \mathcal{M}$. Then, there is a model structure on $\mathcal{M}^{\mathcal{D}}$ whose weak equivalences and fibrations are precisely the pointwise weak equivalences and pointwise fibrations respectively.

To see this, let us denote by $|\mathcal{D}|$ is the (small) set of objects of $\mathcal{D}$, and by $\mathcal{M}^{|\mathcal{D}|}$ the product category of copies of $\mathcal{M}$ indexed by $\mathcal{D}$. Then one has an adjunction

$$\mathbb{F} \,:\, \mathcal{M}^{|\mathcal{D}|} \rightleftarrows \mathcal{M}^{\mathcal{D}} \,:\, U \tag{9.2.6}$$

where U is the forgetful functor that associates a $\mathcal{D}$-diagram in $\mathcal{M}$ its underlying $|\mathcal{D}|$-tuple of objects in $\mathcal{M}$. The left adjoint $\mathbb{F}$ to this forgetful functor has a simple description: for each $|\mathcal{D}|$-tuple $\mathbf{X} = (X_\alpha)_{\alpha \in |\mathcal{D}|}$ of objects in $\mathcal{M}$, the value of $\mathbb{F}(\mathbf{X}) \in \mathcal{M}^{\mathcal{D}}$ on the object $\beta \in \mathcal{D}$ is given by

$$\mathbb{F}(\mathbf{X})(\beta) = \coprod_{\alpha \in |\mathcal{D}|} \coprod_{\mathcal{D}(\alpha,\beta)} X_\alpha$$

To see this, we consider the co-represented functor $h^\alpha : \mathcal{D} \to \mathtt{Set}$ for each $\alpha \in |\mathcal{D}|$. Then the tensoring $\otimes : \mathtt{Set} \times \mathcal{M} \to \mathcal{M}$, given by $S \otimes X := \coprod_S X$, allows one to define $\mathbb{F}_\alpha(X) := h^\alpha \otimes X : \mathcal{D} \to \mathcal{M}$, for each $X \in \mathcal{M}$. Now, for any $G \in \mathcal{M}^{\mathcal{D}}$, we have an adjunction

$$\mathrm{Hom}_{\mathcal{M}^{\mathcal{D}}}(\mathbb{F}_\alpha(X), G) \cong \mathrm{Hom}_{\mathtt{Set}^{\mathcal{D}}}(h^\alpha, \mathrm{Hom}_{\mathcal{M}}(X, G)) \overset{\text{Yoneda}}{\cong} \mathrm{Hom}_{\mathcal{M}}(X, G(\alpha))$$

In other words, a map $\mathbb{F}_\alpha(X) \to G$ in $\mathcal{M}^{\mathcal{D}}$ is equivalent to a map $X \to G(\alpha)$ in $\mathcal{M}$. Therefore, a map from $\mathbb{F}(\mathbf{X}) = \coprod_{\alpha \in |\mathcal{D}|} \mathbb{F}_\alpha(X_\alpha)$ to G is equivalent to a map $X_\alpha \to G(\alpha)$ for each $\alpha \in |\mathcal{D}|$. This shows that (9.2.6) is an adjunction.

Clearly, the product category $\mathcal{M}^{|\mathcal{D}|}$ has a model structure in which weak equivalences, fibrations and cofibrations are defined objectwise. Thus, to give a model structure on $\mathcal{M}^{\mathcal{D}}$, it suffices to have a result that allows one to conclude that a model structure on $\mathcal{M}^{|\mathcal{D}|}$ induces a model structure on $\mathcal{M}^{\mathcal{D}}$ via the forgetful functor $U : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{|\mathcal{D}|}$. There is a general result (see [77, Theorem 11.3.2]) that allows one to transfer model categories in this manner. An application of this result to the adjunctoin (9.2.6) then gives the following

Theorem 9.2.7 ([77], Theorem 11.6.1)**.** *If $\mathcal{D}$ is a small category, and $\mathcal{M}$ is a cofibrantly generated model category with a set I of generating cofibrations and a set J of generating fibrations, then the functor category $\mathcal{M}^{\mathcal{D}}$ has a cofibrantly generated model category, in which*

1. *A map $X \to Y$ is a weak equivalence if it is pointwise weak equivalence;*
2. *A map $X \to Y$ is a fibration if it is pointwise fibration;*
3. *A map $X \to Y$ is a cofibration if it has the left lifting problem with respect to trivial fibrations.*

Moreover, a set of generating cofibrations in $\mathcal{M}^{\mathcal{D}}$ is given by $\bigcup_{\alpha\in|\mathcal{D}|}\mathbb{F}_\alpha(I)$; while a set of generating trivial cofibrations in $\mathcal{M}^{\mathcal{D}}$ is given by $\bigcup_{\alpha\in|\mathcal{D}|}\mathbb{F}_\alpha(J)$.

This model structure is called the *projective model structure*. Notice that every cofibration in the projective model structure is a pointwise cofibration, but the converse is not true in general.

One can apply this theorem to the model category $\mathcal{M} = \mathrm{s}\mathcal{C}$ of simplicial objects on an algebraic category $\mathcal{C}$, and the indexing category $\mathcal{D} = \Delta^{\mathrm{op}}$, in this case, one obtains a (projective) model structure on the category $\mathtt{ss}\mathcal{C}$ of bisimplicial objects on $\mathcal{C}$, in which a map $f : X \to Y$ is a weak equivalences [resp. fibrations] if and only if the induced map $f : X_{p,*} \to Y_{p,*}$ is a weak equivalence [resp. fibration] in $\mathrm{s}\mathcal{C}$ for each fixed external degree p.

With this model structure on $\mathtt{ss}\mathcal{C}$, Proposition 9.2.5 can be viewed as the statement that the diagonal functor (9.2.4) preserves weak equivalences. In fact, the description of generating cofibrations in Theorem 9.2.7 allows us to show that it also preserves cofibrations.

Lemma 9.2.8. *The diagonal functor* (9.2.4) *preserves cofibrations.*

Proof. Both $\mathtt{s}\mathcal{C} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ and $\mathtt{ss}\mathcal{C} = \mathrm{Fun}((\Delta\times\Delta)^{\mathrm{op}}, \mathcal{C})$ are functor categories. Since the algebraic category $\mathcal{C}$ is cocomplete, colimits in both $\mathtt{s}\mathcal{C}$ and $\mathtt{ss}\mathcal{C}$ are computed pointwise. Therefore, the diagonal functor preserves colimits. Thus, to show that it preserves cofibrations, it suffices to show that it sends generating cofibrations in $\mathtt{ss}\mathcal{C}$ to cofibrations in $\mathtt{s}\mathcal{C}$.

Denote by $F : \mathtt{Set} \to \mathcal{C}$ the left adjoint to the forgetful functor $U : \mathcal{C} \to \mathtt{Set}$, then we recall that a set of generating cofibrations for $\mathtt{s}\mathcal{C}$ is given by $I = \{\, F(\partial\Delta^n) \to F(\Delta^n)\,\}_{n\in\mathbb{N}}$. By Theorem 9.2.7, a set of generating cofibrations for $\mathtt{ss}\mathcal{C}$ is then given by

$$\mathbb{I} = \{\ \mathbb{F}_{[m]}(F(\partial\Delta^n)) \to \mathbb{F}_{[m]}(F(\Delta^n))\ \}_{m\in\mathbb{N},n\in\mathbb{N}}$$

Unravelling the relevant definitions, one sees that these are precisely the maps

$$F(\Delta^m \times \partial\Delta^n) \to F(\Delta^m \times \Delta^n)$$

where F is now regarded as a functor $F : \mathtt{ssSet} \to \mathtt{ss}\mathcal{C}$ from bisimplicial sets to bisimplcial objects in $\mathcal{C}$. Therefore, to prove the lemma, it suffices to show that the map

$$F(\mathrm{diag}(\Delta^m \times \partial\Delta^n)) \to F(\mathrm{diag}(\Delta^m \times \Delta^n)) \tag{9.2.9}$$

is a cofibration in $\mathtt{s}\mathcal{C}$. Now, for every simplicial sets X and Y, the diagonal of the bisimplicial set $X\times Y$ is simply the product simplicial set $X\times Y$. Hence, the map $\mathrm{diag}(\Delta^m\times\partial\Delta^n)) \to \mathrm{diag}(\Delta^m\times\Delta^n)$ is a cofibration of simplicial sets. Since F is the left adjoint of the Quillen adjunction (9.1.3), one concludes that the map (9.2.9) is a cofibration in $\mathtt{s}\mathcal{C}$. This completes the proof. □

Remark 9.2.10. The functor $\mathrm{diag} : \mathtt{ss}\mathcal{C} \to \mathtt{s}\mathcal{C}$ has a right adjoint, given by right Kan extension. By Proposition 9.2.5 and Lemma 9.2.8, this adjunction is a Quillen adjunction.

With Proposition 9.2.5 and Lemma 9.2.8, we are now ready to give the following

Proof of Theorem 9.2.2. Let $X \in \mathtt{s}\mathcal{C}$ be a simplicial object in $\mathcal{C}$ such that each $X_n \in \mathcal{C}$ is F-adapted when considered a discrete object in $\mathtt{s}\mathcal{C}$. Define the bisimplicial object $\tilde{X} \in \mathtt{ss}\mathcal{C}$ by $\tilde{X}_{m,n} = X_m$ for all $m, n \in \mathbb{N}$. Thus, $\tilde{X}$ is constant in the internal degree, and equal to X in the external degree. Resolve $\tilde{X}$ by a cofibrant $Q \in \mathtt{ss}\mathcal{C}$, so that we have a weak equivalence $p : Q \xrightarrow{\sim} \tilde{X}$.

Starting with the map $p : Q \xrightarrow{\sim} \tilde{X}$ in $\mathtt{ss}\mathcal{C}$, we will apply the following commutative diagram of functors

$$\begin{tikzcd}
\mathtt{ss}\mathcal{C} \arrow[r, "\mathrm{diag}"] \arrow[d, "F"'] & \mathtt{s}\mathcal{C} \arrow[d, "F"] \\
\mathtt{ss}\mathcal{D} \arrow[r, "\mathrm{diag}"] & \mathtt{s}\mathcal{D}
\end{tikzcd}$$

to produce a map in $\mathtt{s}\mathcal{C}$.

Consider first the image of $p : Q \xrightarrow{\sim} \tilde{X}$ under the upper route. First, applying the diagonal functor gives the map

$$\mathrm{diag}(p) : \mathrm{diag}(Q) \to \mathrm{diag}(\tilde{X}) \tag{9.2.11}$$

Clearly, we have $\mathrm{diag}(\tilde{X}) = X$. Moreover, by Lemma 9.2.8, the object $\mathrm{diag}(Q)$ is cofibrant in $\mathtt{s}\mathcal{C}$. Thus, by Proposition 9.2.5, the map (9.2.11) is a weak equivalence and hence gives a cofibrant replacement of X. Therefore, if we apply F to (9.2.11), the resulting map

$$F(\mathrm{diag}(p)) : F(\mathrm{diag}(Q)) \to F(\mathrm{diag}(\tilde{X})) \tag{9.2.12}$$

represents the canonical map $\boldsymbol{L}F(\gamma(X)) \to \gamma(F(X))$ in the homotopy category $\mathrm{Ho}(\mathtt{s}\mathcal{D})$.

Now, consider the image of $p : Q \xrightarrow{\sim} \tilde{X}$ under the lower route. First, applying F gives the map

$$F(p) : F(Q) \to F(\tilde{X}) \tag{9.2.13}$$

Since Q is cofibrant in the projective model structure on $\mathtt{ss}\mathcal{C}$, it is objectwise cofibrant, *i.e.*, for each fixed external degree p, the simplicial object $Q_{p,*} \in \mathtt{s}\mathcal{C}$ is cofibrant. The assumption that each $X_p \in \mathcal{C}$ is F-adapted when considered a discrete object in $\mathtt{s}\mathcal{C}$ can be rewritten as saying that each $\tilde{X}_{p,*} \in \mathtt{s}\mathcal{C}$ is F-adapted. By definition of F-adaptedness, this implies that the map (9.2.13) is a weak equivalence in $\mathtt{s}\mathcal{D}$ for each fixed external degree. Another application of Proposition 9.2.5 then shows that the map

$$\mathrm{diag}(F(p)) : \mathrm{diag}(F(Q)) \to \mathrm{diag}(F(\tilde{X})) \tag{9.2.14}$$

is a weak equivalence.

To summarize, the above argument shows that the map (9.2.14) is a weak equivalence, while the map (9.2.12) represents the canonical map $\boldsymbol{L}F(\gamma(X)) \to \gamma(F(X))$ in the homotopy category $\mathrm{Ho}(\mathtt{s}\mathcal{D})$. Since the two maps (9.2.12) and (9.2.14) are in fact isomorphic, this shows that the canonical map $\boldsymbol{L}F(\gamma(X)) \to \gamma(F(X))$ is an isomorphism. This is precisely the statement we wish to prove. □

CHAPTER 10

MONOIDAL DOLD-KAN CORRESPONDENCE

The Dold-Kan correspondence is a classical result that establishes an equivalence between the category $\mathtt{Ch}_{\geq 0}(\mathscr{A})$ of non-negatively graded chain complexes in an abelian category $\mathscr{A}$ and the category $s\mathscr{A}$ of simplicial objects in $\mathscr{A}$. In this Section, we will describe a monoidal enrichment of this correspondence relating the category of (nonnegatively graded) DG $\mathcal{P}$-algebras to the category of simplicial $\mathcal{P}$-algebras for an arbitrary k-linear operad $\mathcal{P}$. For simplicity, we will fix a commutative ring k with unit, and consider only the abelian category $\mathscr{A} = \mathtt{Mod}_k$.

10.1 The Dold-Kan correspondence

To any simplicial k-module $X_* \in \mathtt{sMod}_k$ we can associate the chain complex

$$N(X) = [\dots \to N(X)_n \xrightarrow{\partial} N(X)_{n-1} \to \dots]$$

with $N(X)_n := \bigcap_{i=1}^n \mathrm{Ker}(d_i : X_n \to X_{n-1})$ for $n \geq 0$ and the differential ∂ given by d_0. The assignment $X \mapsto N(X)$ defines a functor $N : \mathtt{sMod}_k \to \mathtt{Ch}_{\geq 0}(k)$ from the category of simplicial k-modules to the category of connective chain complexes of k-modules. The functor N is called the *normalization functor*. A classical theorem due to Dold and Kan (see [164, Theorem 8.4.1]) asserts that N is an equivalence of categories.

For any simplicial k-module $X \in \mathtt{sMod}_k$, the homology groups of the chain complex $N(X)$ are naturally isomorphic to the homotopy groups $\pi_*(|X|)$ of the geometric realization of X (see [112, Theorem 22.1]). This justifies the notation $\pi_*(X) := \mathrm{H}_*[N(X)]$, which we use throughout this Part.

The inverse $N^{-1} : \mathtt{Ch}_{\geq 0}(k) \to \mathtt{sMod}_k$ of the normalization functor is defined as follows. For any chain complex $V \in \mathtt{Ch}_{\geq 0}(k)$, the degree n part of the simplicial k-module $N^{-1}(V)$ is given by

$$N^{-1}(V)_n = \bigoplus_{r \geq 0} \bigoplus_{\sigma : [n] \twoheadrightarrow [r]} V_r \tag{10.1.1}$$

We think of $N^{-1}(V)$ as adjoining to V the degeneracies of all elements in V. We write an element $x \in V_r$ in the summand corresponding to σ as $\sigma^*(x)$. When $\sigma = \mathrm{id}$, we simply write this as x, or $\eta(x)$ if we want to emphasize that we consider x to be an element in $N^{-1}(V)$ rather than V. As suggested by the notation, this determines the degeneracy maps in $N^{-1}(V)$: namely, $s_j(\sigma^*(x)) := (\sigma \circ \sigma^j)^*(x)$ The face maps in $N^{-1}(V)$ are determined by the requirement that $d_i(\eta(x)) = 0$ for all $i > 0$, and the canonical map

$$\eta : V \to N[N^{-1}(V)], \qquad\qquad x \mapsto \eta(x) = x$$

commutes with differentials, *i.e.* $d_0(\eta(x)) = \eta(d(x))$. Since all elements of $N^{-1}(V)$ other than $\eta(x)$ are sums of degenerations of $\eta(x)$, specifying the face maps on these elements determines all the face maps in $N^{-1}(V)$. This defines a simplicial k-module $N^{-1}(V)$ and hence the functor $N^{-1} : \mathtt{Ch}_{\geq 0}(k) \to \mathtt{sMod}_k$ (see [67] for more details). It is easy to check that this functor is indeed the inverse of the normalization functor N.

There is an alternative way to define the normalization functor. For each simplicial k-module $X \in \mathtt{sMod}_k$, we can take the chain complex $\overline{N}(X)$ defined by

$$\overline{N}(X)_n := \frac{X_n}{\sum_{j=0}^{n-1} s_j(X_{n-1})} \qquad\qquad d = \sum_{i=0}^{n} (-1)^i d_i : \overline{N}(X)_n \to \overline{N}(X)_{n-1} \tag{10.1.2}$$

Then one can show (see [67]) that the canonical map $N(X) \to \overline{N}(X)$ of chain complexes given by the composition $N(X)_n \hookrightarrow X_n \twoheadrightarrow \overline{N}(X)_n$ is an isomorphism.

Notice that the inverse (10.1.1) of the normalization functor has an important feature: the collection of k-modules $N^{-1}(V)_n$, as well as the degeneracy maps between them, depends only on the graded module V and not on its differential. In other words, (10.1.1) defines a functor $N^{-1} : \mathtt{grMod}_k \to \mathtt{Mod}_k^{\Delta^{\mathrm{op}}_{\mathrm{surj}}}$ from the category $\mathtt{grMod}_k$ of graded k-modules to the category $\mathtt{Mod}_k^{\Delta^{\mathrm{op}}_{\mathrm{surj}}}$ of $\Delta^{\mathrm{op}}_{\mathrm{surj}}$-systems of k-modules. Similarly, (10.1.2) gives a functor $\overline{N} : \mathtt{Mod}_k^{\Delta^{\mathrm{op}}_{\mathrm{surj}}} \to \mathtt{grMod}_k$. This will play a role in our construction of the monoidal Dold-Kan correspondence in the next section.

10.2 Monoidal Dold-Kan correspondence

It is a classical fact that the Dold-Kan normalization functor $N : \mathtt{sMod}_k \to \mathtt{Ch}_{\geq 0}(k)$ can be endowed with a symmetric lax monoidal structure. To describe it, we first introduce some notations. Given two simplicial modules $X, Y \in \mathtt{sMod}_k$ over a commutative ring k, we denote by $X \mathbin{\bar{\otimes}} Y \in \mathtt{sMod}_k$ the result of applying the tensor product levelwise, *i.e.*, $(X \mathbin{\bar{\otimes}} Y)_n := X_n \otimes_k Y_n$. Then, there is a *quasi-isomorphism* of chain complexes

$$\mathrm{sh} : N(X) \otimes N(Y) \xrightarrow{\sim} N(X \mathbin{\bar{\otimes}} Y)$$

called the *Eilenberg-Zilber shuffle map*, which is natural (in X and Y), symmetric, associative and unital in the obvious sense (see, e.g., [112, 145] for details).

This shuffle map allows one to transfer algebraic structures from a simplicial module A to its normalization $N(A)$. For instance, if A is a simplicial associative algebra, then $N(A)$ is a DG algebra; if A is a simplicial commutative algebra, then $N(A)$ is a commutative DG algebra, *etc*. In general, for any k-linear operad $\mathcal{P}$, one can consider the category $\mathtt{sAlg}(\mathcal{P})$ of simplicial $\mathcal{P}$-algebras as well as the category $\mathtt{dgAlg}(\mathcal{P})$ of non-negatively graded DG $\mathcal{P}$-algebras. If $A \in \mathtt{sAlg}(\mathcal{P})$ is a simplicial $\mathcal{P}$-algebra, then each n-ary operation $\mu \in \mathcal{P}(n)$ gives a map

$$\alpha_A(\mu) \,:\, A \,\bar{\otimes}\, \overset{n}{\ldots} \,\bar{\otimes}\, A \,\to\, A$$

One can then use the Eilenberg-Zilber shuffle maps to construct the maps

$$\alpha_{N(A)}(\mu) \,:\, N(A) \otimes \overset{n}{\ldots} \otimes N(A) \xrightarrow{\mathrm{sh}} N(\,A \,\bar{\otimes}\, \overset{n}{\ldots} \,\bar{\otimes}\, A\,) \xrightarrow{N(\alpha_A(\mu))} N(A)$$

which form the structure maps for a DG $\mathcal{P}$-algebra on $N(A)$. This defines a functor

$$N : \mathtt{sAlg}(\mathcal{P}) \,\to\, \mathtt{dgAlg}(\mathcal{P})\,. \tag{10.2.1}$$

In the special case when $\mathcal{P}$ is the Lie operad, this last functor has already appeared in [136]. Quillen showed that it has a left adjoint in that case. His proof generalizes directly to an arbitrary operad.

Proposition 10.2.2. *The functor* (10.2.1) *has a left adjoint* $N^* \,:\, \mathtt{dgAlg}(\mathcal{P}) \,\to\, \mathtt{sAlg}(\mathcal{P})$*.*

Proof. As in [136], for any $A \in \mathtt{dgAlg}(\mathcal{P})$, we define $N^*(A)$ as the following (degreewise) coequalizer of simplicial $\mathcal{P}$-algebras

$$N^*(A) \,=\, \mathrm{coeq}\,\left[\, T_{\mathcal{P}}(N^{-1}(T_{\mathcal{P}}(A))) \underset{\mathrm{sh}_*}{\overset{\alpha_*}{\rightrightarrows}} T_{\mathcal{P}}(N^{-1}(A)) \,\right]$$

where α_* and sh_* are induced by $N^{-1}(\alpha) \,:\, N^{-1}(T_{\mathcal{P}}(A)) \,\to\, N^{-1}(A)$ and the Eilenberg-Zilber maps $\mathrm{sh} \,:\, N^{-1}(T_{\mathcal{P}}(A)) \,\to\, T_{\mathcal{P}}(N^{-1}(A))$ respectively. $\square$

Notice that the proof shows that the underlying $\Delta^{\mathrm{op}}_{\mathrm{surj}}$-system of $\mathcal{P}$-algebra of $N^*(A)$ depends only on the graded algebra structure of A (see the discussion at the end of the previous subsection). This observation will allow us to describe the simplicial $\mathcal{P}$-algebra $N^*(A)$ in the case when A is semi-free.

We first consider the commutative diagrams of functors

$$\begin{tikzcd} \mathtt{sAlg}(\mathcal{P}) \arrow[r, "\text{forget}"] \arrow[d, "N"'] \arrow[dr, phantom, "(1)"] & \mathtt{sMod}_k \arrow[d, "N"] \\ \mathtt{dgAlg}(\mathcal{P}) \arrow[r, "\text{forget}"] & \mathtt{Ch}_{\geq 0}(k) \end{tikzcd} \qquad \begin{tikzcd} \mathtt{sAlg}(\mathcal{P}) \arrow[dr, phantom, "(2)"] & \mathtt{sMod}_k \arrow[l, "T_{\mathcal{P}}"'] \\ \mathtt{dgAlg}(\mathcal{P}) \arrow[u, "N^*"] & \mathtt{Ch}_{\geq 0}(k) \arrow[l, "T_{\mathcal{P}}"'] \arrow[u, "N^{-1}"'] \end{tikzcd}$$

where we denote by $T_{\mathcal{P}}$ the free algebra functors in both simplicial and DG contexts. The square (1) obviously commutes up to isomorphism of functors. The square (2) is obtained by replacing every functor on the square (1) by its left adjoint. Therefore, it also commutes up to isomorphism of functors. The commutativity (up to isomorphism) of the square (2) can be written as

$$N^*(T_{\mathcal{P}}(V)) \cong T_{\mathcal{P}}(N^{-1}(V)) \ . \tag{10.2.3}$$

In other words, N^* of a free DG $\mathcal{P}$-algebra is free. The same is true for semi-free algebras. Recall that a DG $\mathcal{P}$-algebra is said to be *semi-free* if its underlying graded algebra is free over a degreewise free graded k-module V. Similarly, a simplicial $\mathcal{P}$-algebra A is said to be *semi-free*[1] if its underlying $\Delta^{\mathrm{op}}_{\mathrm{surj}}$-system of $\mathcal{P}$-algebras is of the form $A = T_{\mathcal{P}}(N^{-1}(V))$ for a degreewise free graded k-module V.

The above discussion lreads to the following

[1]By standard definition (cf. [67]), a simplicial $\mathcal{P}$-algebra is called semi-free if there is a collection of subsets $B_n \subset A_n$, called a *basis*, that is closed under degeneracies and that $A_n = T_{\mathcal{P}}(B_n)$ for each n. It is clear that our definition implies this. To see the converse, notice that any basis element that is not the degeneracy of any other basis element is in fact non-degenerate in the underlying simplicial set of A. Let V be the graded k-module with a basis given by these non-degenerate basis elements. Then an application of [70, Lemma I.2.11] shows that $A = T_{\mathcal{P}}(N^{-1}(V))$ as a $\Delta^{\mathrm{op}}_{\mathrm{surj}}$-system of $\mathcal{P}$-algebras.

Proposition 10.2.4. *The functor $N^* : \mathtt{dgAlg}(\mathcal{P}) \to \mathtt{sAlg}(\mathcal{P})$ sends semi-free DG $\mathcal{P}$-algebras to semi-free simplicial $\mathcal{P}$-algebras.*

Proof. We have seen that N^* sends free algebras to free algebras. Since the underlying $\Delta^{\mathrm{op}}_{\mathrm{surj}}$-system of $N^*(A)$ depends only on the graded algebra structure of A, the result follows. $\square$

Next, we consider the adjunction map $A \to N(N^*(A))$ in the case when $A = T_{\mathcal{P}}(V)$ is semi-free over a graded complex V. To describe this map, we first give a different interpretation of the Eilenberg-Zilber shuffle map. Namely, we view it a collection of maps that connect two symmetric monoidal structures on the category $\mathtt{Ch}_{\geq 0}(k)$ of chain complexes on k. We will use the "quotient" form (10.1.2) of the normalization functor. Thus, we consider the equivalence of categories $\overline{N} : \mathtt{sMod}_k \to \mathtt{Ch}_{\geq 0}(k)$. One can use this equivalence to transport the symmetric monoidal structure $\bar{\otimes}$ on $\mathtt{sMod}_k$ to a symmetric monoidal structure $\underline{\otimes}$ on $\mathtt{Ch}_{\geq 0}(k)$. Precisely, we define $V \underline{\otimes} W := \overline{N}(N^{-1}(V) \bar{\otimes} N^{-1}(W))$ for $V, W \in \mathtt{Ch}_{\geq 0}(k)$. Then the Eilenberg-Zilber shuffle maps can be written as

$$\mathrm{sh} : V \otimes W \to V \underline{\otimes} W, \qquad x \otimes y \mapsto x \times y := \mathrm{sh}(x, y) \tag{10.2.5}$$

Now, suppose that a DG $\mathcal{P}$-algebra $A \in \mathtt{dgAlg}(\mathcal{P})$ is semi-free over a graded k-module V, *i.e.*,

$$A = T_{\mathcal{P}}(V) := \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$$

then by (10.2.3) as well as the discussion that follows, the DG $\mathcal{P}$-algebra $\overline{N}(N^*(A)) \in \mathtt{dgAlg}(\mathcal{P})$ has a similar description

$$\overline{N}(N^*(A)) = \bigoplus_{n \geq 0} \mathcal{P}(n) \underline{\otimes}_{S_n} V^{\underline{\otimes} n}$$

Moreover, the adjunction map $A \to \overline{N}(N^*(A))$ is given by

$$\bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n} \to \bigoplus_{n\geq 0} \mathcal{P}(n) \underline{\otimes}_{S_n} V^{\underline{\otimes} n}, \qquad (\mu\,,\, x_1 \otimes \ldots \otimes x_n) \mapsto (\mu\,,\, x_1 \times \ldots \times x_n) \tag{10.2.6}$$

This description of the adjunction map will be useful in the next subsection when we compare the model structures on simplicial $\mathcal{P}$-algebras and DG $\mathcal{P}$-algebras.

10.3 Quillen equivalence

By [135, Section II.4, Theorem 4], there is a model structure on the category $\mathtt{sAlg}(\mathcal{P})$ of simplicial $\mathcal{P}$-algebras where a map $f : A \to B$ is a weak equivalence (resp., fibration) if and only if the map of the underlying simplicial sets is a weak equivalence (resp., fibration). Moreover, it is shown in [78] that if k is a field of characteristic 0, then the category $\mathtt{dgAlg}(\mathcal{P})$ of DG $\mathcal{P}$-algebras also has a model structure in which a map $f : A \to B$ is a weak equivalence (resp., fibration) if and only if the map of the underlying (connective) chain complexes is a weak equivalence (resp., fibration).

From now on, we assume that k is a field of characteristic 0, and the categories $\mathtt{sAlg}(\mathcal{P})$ and $\mathtt{dgAlg}(\mathcal{P})$ are equipped with the model structures described above. Then the normalization functor $N : \mathtt{sAlg}(\mathcal{P}) \to \mathtt{dgAlg}(\mathcal{P})$ preserves fibrations and weak equivalences. Therefore, the adjunction

$$N^* \,:\, \mathtt{dgAlg}(\mathcal{P}) \rightleftarrows \mathtt{sAlg}(\mathcal{P}) \,:\, N \tag{10.3.1}$$

is a Quillen pair. In fact, we have the following theorem, which is the main result of this section.

Theorem 10.3.2. *The Quillen pair* (10.3.1) *is a Quillen equivalence.*

Proof. It suffices to show that, for any semi-free DG $\mathcal{P}$-algebra $A = T_{\mathcal{P}}(V)$, unit of the adjunction (10.3.1) is a weak equivalence. Composing this adjunction map with the isomorphism $N(N^*(A)) \to \overline{N}(N^*(A))$, we can consider the map $A \to \overline{N}(N^*(A))$, which depends only on the underlying graded $\mathcal{P}$-algebra structure of A, and is described explicitly by (10.2.6)

If A is free, *i.e.*, when the differential on $A = T_{\mathcal{P}}(V)$ is induced by the differential on a chain complex V, then for each $n \geq 0$, the map

$$\mathrm{sh} : \mathcal{P}(n) \otimes V^{\otimes n} \to \mathcal{P}(n) \underline{\otimes} V^{\underline{\otimes} n}$$

is a quasi-isomorphism because it is induced by the Eilenberg-Zilber shuffle map (which is always a quasi-isomorphism). Since k is a field of characteristic 0, the same is true when we pass to S_n-coinvariants

$$\mathrm{sh} : \mathcal{P}(n) \otimes_{S_n} V^{\otimes n} \to \mathcal{P}(n) \underline{\otimes}_{S_n} V^{\underline{\otimes} n}$$

This shows that the map (10.2.6) is a quasi-isomorphism in the case when A is free.

In the general case, when $A = T_{\mathcal{P}}(V)$ is semi-free over a graded k-module V, choose a homogeneous basis of V, and assign a weight grading $\mathrm{wt}(x) \in \mathbb{N}$ for each such basis element x. This induces a grading on A, where an element $(\mu\, ,\, x_1 \otimes \ldots \otimes x_n) \in \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$ has weight grading $\mathrm{wt}(x_1) + \cdots + \mathrm{wt}(x_n)$.

The underlying $\Delta^{\mathrm{op}}_{\mathrm{surj}}$-system of $N^*(A)$ is given by $N^*(A) = T_{\mathcal{P}}(N^{-1}(V))$. Therefore, its elements in degree m are sums of elements of the form $(\mu\, ,\, \sigma_1^*(x_1) \otimes \ldots \otimes \sigma_n^*(x_n))$ where $\phi_i : [m] \twoheadrightarrow [r_i]$ and $x_i \in V_{r_i}$. Assign the weight grading $\mathrm{wt}(x_1) + \cdots + \mathrm{wt}(x_n)$ to this element. Then it is clear that all the degeneracy maps preserve this weight grading. This induces a weight grading in the normalization $\overline{N}(N^*(A))$. Moreover, the map (10.2.6) preserves this grading. We write this grading as

$$A = \bigoplus_{n\geq 0} A^{(n)}\,, \qquad N^*(A) = \bigoplus_{n\geq 0} N^*(A)^{(n)}\,, \qquad \overline{N}(N^*(A)) = \bigoplus_{n\geq 0} \overline{N}(N^*(A))^{(n)} \tag{10.3.3}$$

In general, the differentials on both sides of (10.2.6) do not preserve the grading. However, if we let $F_n(A) = \bigoplus_{i\leq n} A^{(i)}$ be the filtration on A induced by the weight grading, then we can always choose the weight grading on a homogeneous basis of V inductively so that $d(F_n(A)) \subset F_{n-1}(A)$. Moreover, if we let $G_n = G_n(\overline{N}(N^*(A))) = \bigoplus_{i\leq n} \overline{N}(N^*(A))^{(i)}$ be the filtration on $\overline{N}(N^*(A))$ induced by the weight grading on $\overline{N}(N^*(A))$, then we claim that $d(G_n) \subset G_n$ for all n.

Indeed, consider the filtration $\tilde{G}_n = \bigoplus_{i\leq n} N^*(A)^{(i)}$ on $N^*(A)$ induced by the weight grading. As we have seen, each graded piece $N^*(A)^{(i)}$ is a $\Delta^{\mathrm{op}}_{\mathrm{surj}}$-system of k-modules such that $\overline{N}(N^*(A)^{(i)}) = \overline{N}(N^*(A)^{(i)})$. Therefore, to show that $d(G_n) \subset G_n$, it suffices to show that $d_i(\tilde{G}_n) \subset \tilde{G}_n$ for all face maps d_i. We will in fact show a more refined statement. To express this statement, we recall that the face maps of $N^*(A)$ are determined by the fact that the adjunction map $\eta : A \to N(N^*(A))$ commutes with the differential. Indeed, for each homogeneous basis element $x \in V$, considered as an element in $T_{\mathcal{P}}(V) = A$, the requirements $d_0(\eta(x)) = d(\eta(x)) = \eta(d(x))$ and $d_i(\eta(x)) = 0$ specify the values of face maps on the non-degenerate generators $\eta(x)$ of $N^*(A) = T_{\mathcal{P}}(N^{-1}(V))$. This in turn specifies the face maps on every other elements by simplicial identites. Thus, one can write the face maps as $d_i = d_i[d_A]$ to show its dependence on the differential d_A on A. In Lemma 10.3.5 below, we will show that, for any differential $d = d_A$ on A such that $d(F_n) \subset F_{n-1}$, the face maps

d_i when restricted to homogeneous elements $z \in N^*(A)^{(n)}$, can be decomposed as $d_i[d_A](z) = d_i'(z) + d_i''[d_A](z)$, where $d_i' : N^*(A)^{(n)} \to N^*(A)^{(n)}$ and $d_i''[d_A] : N^*(A)^{(n)} \to \tilde{G}_{n-1}$. Moreover, d_i' does not depend on the differential d_A, and $d_i''[d_A] = 0$ if $d_A = 0$. In particular, we have $d_i' = d_i[0]$.

Assuming this lemma, then we have $d_i(\tilde{G}_n) \subset \tilde{G}_n$, and hence $d(G_n) \subset G_n$. Therefore, both the domain and target of the map (10.2.6) of chain complexes admit filtrations by subcomplexes, such that the map (10.2.6) preserves these subcomplexes. Since these filtrations are induced by gradings, the graded k-modules associated to these filtrations can be canonically identified with the original graded k-modules, *i.e.,* we have

$$\begin{aligned} \mathrm{gr}_F(A) &\cong \bigoplus_{n\geq 0} A^{(n)} = A \\ \mathrm{gr}_G(\overline{N}(N^*(A))) &\cong \bigoplus_{n\geq 0} \overline{N}(N^*(A))^{(n)} = \overline{N}(N^*(A)) \end{aligned} \tag{10.3.4}$$

as graded k-modules. While passing to the associated graded modules does not change the underlying graded k-modules, it changes the differentials by discarding the part that strictly decrease the grading. Since we have chosen the differential d on A such that $d(F_n) \subset F_{n-1}$, the induced differential on $\mathrm{gr}_F(A)$ is zero. In other words, (10.3.4) actually identifies $\mathrm{gr}_F(A)$ with the free DG $\mathcal{P}$ algebra $A' = T_{\mathcal{P}}(V)$ with trivial differential. On the other hand, our above discussion, expressed in Lemma 10.3.5 below, gives a description of the differential on the associated graded k-module $\mathrm{gr}_G(\overline{N}(N^*(A)))$. Namely, by discarding the part of the differential on $\overline{N}(N^*(A))$ that strictly decreases the grading, one retains precisely the differential in $\overline{N}(N^*(A'))$ where A' is again the free DG $\mathcal{P}$ algebra $A' = T_{\mathcal{P}}(V)$ with trivial differential. In other words, (10.3.4) actually identifies $\mathrm{gr}_G(\overline{N}(N^*(A)))$ with $\overline{N}(N^*(A'))$.

Therefore, the induced map $\mathrm{gr}_F(A) \to \mathrm{gr}_G(\overline{N}(N^{-1}(A)))$ on the associated graded chain complexes coincides with the adjunction map $A' \to \overline{N}(N^*(A'))$ for the free algebra $A' = T_{\mathcal{P}}(V)$ with zero differential. This map is a quasi-isomorphism by our previous argument in the free case. Since the filtrations $F_\bullet$ and $G_\bullet$ are bounded below and exhaustive, the map (10.2.6) induces an isomorphism on homology by the Eilenberg-Moore comparison theorem [164, Thm. 5.5.11]. $\square$

Lemma 10.3.5. *For any differential $d = d_A$ on A such that $d(F_n) \subset F_{n-1}$, let $d_i = d_i[d_A]$ be the i-th face maps on $N^*(A)$. Then its restriction $d_i|_{N^*(A)^{(n)}}$ to each homogeneous component $N^*(A)^{(n)}$ can be decomposed as $d_i[d_A](z) = d_i'(z) + d_i''[d_A](z)$, where $d_i' : N^*(A)^{(n)} \to N^*(A)^{(n)}$ and $d_i''[d_A] : N^*(A)^{(n)} \to \tilde{G}_{n-1}$. Moreover, d_i' does not depend on the differential d_A, and $d_i''[d_A] = 0$ if $d_A = 0$.*

Proof. In simplicial degree m, the k-module $N^*(A)^{(n)}_m$ consists of sums of elements of the form

$$z = (\mu\, ,\, \sigma_1^*(x_1) \otimes \ldots \otimes \sigma_k^*(x_k))$$

with $\mathrm{wt}(x_1) + \cdots + \mathrm{wt}(x_k) = n$, where $x_j \in V_{r_j}$ and σ_j are surjective maps $[m] \twoheadrightarrow [r_j]$. The image under the face map d_i of this element is given by

$$d_i(z) \;=\; (\mu\, ,\, d_i(\sigma_1^*(x_1)) \otimes \ldots \otimes d_i(\sigma_k^*(x_k))) \tag{10.3.6}$$

Now, for each $j = 1, \ldots, k$, the element $d_i(\sigma_j^*(x_j))$ reduces by simplicial identities to either of the two cases:

(I) $d_i(\sigma_j^*(x_j)) = \sigma_j'^*(x_j)$ for some surjective map $\sigma_j' : [m-1] \twoheadrightarrow [r_j]$ in Δ.

(II) $d_i(\sigma_j^*(x_j)) = \sigma_j'^*(d_{i'}(x_j)))$ for some surjective map $\sigma_j' : [m-1] \twoheadrightarrow [r_j - 1]$ in Δ, and some $0 \le i' \le r_j$.

In case (I), $d_i(\sigma_j^*(x_j))$ has the same weight grading as the term $\sigma_j^*(x_j)$. We split the case (II) in two subcases:

(IIa) If $i' > 0$, then we have $d_{i'}(x_j) = 0$ because by definition $x_j = \eta(x_j)$ is in $N(N^*(A))$.

(IIb) If $i' = 0$, then we claim that $d_0(x_j) \in \tilde{G}_{r_j-1}$, and $d_0(x_j) = 0$ if $d_A = 0$.

Indeed, the 0-th face $d_0(x_j)$ of $x_j = \eta(x_j)$ is uniquely determined by the corresponding differential $d(x_j)$ in the DG $\mathcal{P}$-algebra A. Namely, since the adjunction map $\eta : A \to N(N^*(A))$ commutes with differentials, we have $d_0(\eta(x_j)) = \eta(d(x_j))$. Since we have chosen the weight grading on the generators x_j in such a way that $d(x_j)$ is sum of terms of weight grading strictly less than x_j, we see that $d_0(x_j) \in \tilde{G}_{r_j-1}$ in this case. The equation $d_0(\eta(x_j)) = \eta(d(x_j))$ also shows that $d_0(x_j) = 0$ if $d_A = 0$.

Thus, to compute $d_i(z)$, one combines the equation (10.3.6) with the above consideration. If we are in case (I) or (IIa) for each $1 \leq j \leq k$, then $d_i(z)$ is still in $N^*(A)^{(n)}$. Thus, we have $d_i(z) = d_i'(z)$ in this case. Moreover, our explicit description shows that d_i' does not depend on d_A. If we are in case (IIb) for some $1 \leq j \leq k$, then we have $d_i(z) \in \tilde{G}_{n-1}$. Thus, we have $d_i(z) = d_i''(z)$ in this case. Moreover, our description shows that $d_i(z) = 0$ in this case if $d_A = 0$. This completes the proof of the lemma. $\square$

CHAPTER 11

SIMPLICIAL COMMUTATIVE ALGEBRAS

Fix a field k of characteristic 0. By Theorem 10.3.2, there is a Quillen equivalence between the model category $\mathtt{sCommAlg}_k$ of simplicial commutative algebras and the model category $\mathtt{DGCA}_k^+$ of connective commutative differential graded algebras:

$$N^* : \mathtt{DGCA}_k^+ \rightleftarrows \mathtt{sCommAlg}_k : N \tag{11.0.1}$$

This equivalence allows one to pass between commutative DG algebras and simplicial algebras. We first investigate a finite-type condition.

11.1 Simplicial commutative algebras of quasi-finite type

The goal of this subsection is to show the following

Proposition 11.1.1. *For an object $[A] \in Ho(\mathtt{sCommAlg}_k)$, the following conditions are equivalent:*

(1) *$\pi_0(A)$ is a finitely generated k-algebra, and each $\pi_n(A)$ is finitely generated as a module over $\pi_0(A)$.*

(2) *The homotopy type $N([A]) \in Ho(\mathtt{DGCA}_k^+)$ has a representative $B \in \mathtt{DGCA}_k^+$ such that B is semi-free DG algebra over k with finitely many generators in each homological degree.*

(3) *The homotopy type $[A]$ has a representative $A = \{A_n\}_{n\geq 0}$ in $\mathtt{sCommAlg}_k$ such that each A_n is finitely generated over k.*

Definition 11.1.2. An object $[A] \in \mathrm{Ho}(\mathtt{sCommAlg}_k)$ that satisfies the equivalent conditions of Proposition 11.1.1 is called an algebra of *quasi-finite type*.

We will first prove the equivalence between (1) and (2). Since $\pi_n(A) = H_n(N(A))$ for any $A \in \texttt{sCommAlg}_k$, the implication (2) $\Rightarrow$ (1) is clear. The implication (2) $\Rightarrow$ (1) follows from the following

Lemma 11.1.3. *Given any map $f : A \to B$ in $\texttt{DGCA}_k^+$ such that*

1. *The commutative DG algebra A is generated by finitely many generators in each degree.*
2. *The commutative algebra $H_0(B)$ is a finitely generated algebra and each $H_n(B)$ is finitely generated as a module over $H_0(B)$.*

Then the map f factors as

$$A \overset{g}{\hookrightarrow} \tilde{B} \underset{\sim}{\overset{\phi}{\to}} B$$

where g is a semi-free extension with finitely many generators in each degree, and ϕ is a weak equivalence.

Proof. We shall construct by induction a sequence of semi-free extension

$$\begin{tikzcd}
A = \mathrm{sk}_{-1}(g) \arrow[r, hook] \arrow[drr, "\phi^{(-1)}=f"'] & \mathrm{sk}_0(g) \arrow[r, hook] \arrow[dr, "\phi^{(0)}"] & \mathrm{sk}_1(g) \arrow[r, hook] \arrow[d, "\phi^{(1)}"] & \dots \\
 & & B &
\end{tikzcd}$$

where $\mathrm{sk}_n(g)$ is a semi-free extension of $\mathrm{sk}_{n-1}(g)$ of finitely many generators in degree n, together with $\phi^{(n)} : \mathrm{sk}_n(g) \to B$ in $\texttt{DGCA}_k^+$ that extends $\phi^{(n-1)}$, and induces bijection $H_p(\mathrm{sk}_n(g)) \to H_p(B)$ for each $p < n$ as well as surjection $H_n(\mathrm{sk}_n(g)) \to H_n(B)$.

The pair $(\mathrm{sk}_0(g), \phi^{(0)})$ is simple to construct. Namely, since $H_0(B)$ is finitely generated, choose $b_1^{(0)}, ..., b_{r_0}^{(0)} \in B_0$ whose image in $H_0(B)$ generate $H_0(B)$ over $f(H_0(A)) \subset H_0(B)$. Then, define $sk_0(g)$ to be the tensor product $sk_0(g) = A \otimes$

$k[b_1^{(0)}, ..., b_{r_0}^{(0)}]$ and $\phi^{(0)} : \mathrm{sk}_0(g) \to B$ to be the unique map of commutative DG algebras that extend $\phi^{(-1)} = f$ and maps $x_i^{(0)}$ to $b_i^{(0)} \in B_0$. Clearly, $H_0(sk_0(g)) \to H_0(B)$ is surjective.

Suppose for some $n \geq 1$ that $sk_{n-1}(g)$ and $\phi^{(n-1)} : sk_{n-1}(g) \to B$ has been constructed that satisfies the conditions as described above. In particular, $\mathrm{sk}_{n-1}(g)$ is generated by finitely many generators in each degree. In order to construct $(\mathrm{sk}_n(g), \phi^{(n)})$, we need to add varaibles $y_1^{(n)}, \ldots, y_{s_n}^{(n)}$ to kill off the kernel of $\phi^{(n-1)} : H_{n-1}(\mathrm{sk}_{n-1}(g)) \twoheadrightarrow H_{n-1}(B)$, so as to ensure a bijection $\phi^{(n)} : H_{n-1}(\mathrm{sk}_n(g)) \xrightarrow{\sim} H_{n-1}(B)$; as well as variables $x_1^{(n)}, \ldots, x_{r_n}^{(n)}$ to ensure a surjection $\phi^{(n)} : H_n(\mathrm{sk}_n(g)) \twoheadrightarrow H_n(B)$.

Thus, consider the kernel $K^{(n-1)}$ of $\phi^{(n-1)} : H_{n-1}(\mathrm{sk}_{n-1}(g)) \twoheadrightarrow H_{n-1}(B)$. Since $H_{n-1}(\mathrm{sk}_{n-1}(g))$ is finitely generated over the Noetherian ring $H_0(\mathrm{sk}_{n-1}(g))$, the submodule $K^{(n-1)}$ is also finitely generated over $H_0(\mathrm{sk}_{n-1}(g))$. Let $z_1^{(n-1)}, \ldots, z_{s_n}^{(n-1)}$ be elements of $Z_{n-1}(sk_{n-1}(g))$ whose image in $H_{n-1}(sk_{n-1}(g))$ generate $K^{(n-1)}$ as a $H_0(sk_{n-1}(g))$-module. We will add variables $y_1^{(n)}, \ldots, y_{s_n}^{(n)}$ to $sk_{n-1}(g)$ whose differentials are $z_1^{(n-1)}, \ldots, z_{s_n}^{(n-1)}$, so as to "kill off" $K^{(n-1)}$.

Similarly, since $H_n(B)$ is finitely generated as a module over $H_0(B)$, one can find elements $b_1^{(n)}, \ldots, b_{r_n}^{(n)} \in Z_n(B)$ whose image in $H_n(B)$ generate $H_n(B)$ over $H_0(B)$. We will add varaibles $x_1^{(n)}, \ldots, x_{r_n}^{(n)}$ to $sk_{n-1}(g)$ whose differentials are zero, so as to ensure surjectivity $\phi^{(n)} : H_n(\mathrm{sk}_n(g)) \twoheadrightarrow H_n(B)$ of the extension $(\mathrm{sk}_n(g), \phi^{(n)})$. Thus, we define

$$sk_n(g) = sk_{n-1}(g) \otimes k[\, y_1^{(n)}, \ldots, y_{s_n}^{(n)}, x_1^{(n)}, \ldots, x_{r_n}^{(n)} \,]$$

with differentials $d(y_i^{(n)}) = z_i^{(n-1)}$ and $d(x_i^{(n)}) = 0$.

To define the extension $\phi^{(n)} : \mathrm{sk}_n(g) \to B$, it suffices to specify the images of the new variables $y_1^{(n)}, \ldots, y_{s_n}^{(n)}, x_1^{(n)}, \ldots, x_{r_n}^{(n)}$. The variables $x_i^{(n)}$ are simply sent to $b_i^{(n)} \in Z_n(B)$. To specify the image of the variables $y_i^{(n)}$, we recall that the classes $[z_1^{(n-1)}] \in H_{n-1}(\mathrm{sk}_{n-1}(g))$ are sent to zero by $\phi^{(n-1)} : H_{n-1}(\mathrm{sk}_{n-1}(g)) \twoheadrightarrow H_{n-1}(B)$. Thus, there exists elements $c_1^{(n)}, ..., c_{s_{n-1}}^{(n)} \in B_n$ such that $d_B(c_i^{(n)}) = \phi^{(n-1)}(z_i^{(n-1)})$. This allows us to define

$$\phi^{(n)} : \mathrm{sk}_n(g) \ \to \ B, \qquad\qquad x_i^{(n)} \mapsto b_i^{(n)} \qquad\quad y_i^{(n)} \mapsto c_i^{(n)}$$

By construction, this map commutes with differentials, and induces a bijection $\phi^{(n)} : H_{n-1}(\mathrm{sk}_n(g)) \xrightarrow{\sim} H_{n-1}(B)$ as well as a surjection $\phi^{(n)} : H_n(\mathrm{sk}_n(g)) \twoheadrightarrow H_n(B)$, thus finishing the induction. $\square$

Now, we turn to the proof of the equivalence between (2) and (3). The direction $(2) \Rightarrow (3)$ follows from the Quillen equivalence (11.0.1). Indeed, if B is a semi-free representative of $N([A])$, then the Quillen equivalence (11.0.1) implies that $N^*(B)$ is a representative of $[A]$. Since B is furthermore assumed to have finitely many generators in each degree, $N^*(B)$ is also finitely generated in each simplicial degree. To prove the converse $(3) \Rightarrow (2)$, we apply the following lemma, which proof uses an inductive construction similar to, but more subtle than, that of Lemma 11.1.3.

Lemma 11.1.4. *Given $D \in \mathtt{DGCA}_k^+$ and $A \in \mathtt{sCommAlg}_k$, such that D is semi-free with finitely many generators in each homological degree, and A_p is finitely generated over k for each simplicial degree p. Then any map $f : N^*(D) \to A$ of simplicial commutative algebras factors as*

$$N^*(D) \xhookrightarrow{N^*(g)} N^*(B) \underset{\sim}{\xrightarrow{\phi}} A$$

where $g : D \hookrightarrow B$ is a semi-free extension of commutative DG algebras with finitely many generators in each degree, and $\phi : N^*(B) \xrightarrow{\sim} A$ is a surjective weak equivalence of simplicial commutative algebras.

Proof. We will construct such a semi-free extension $g : D \hookrightarrow B$ by inductively constructing a sequence of semi-free extensions of commutative DG algebras:

$$D = \mathrm{sk}_{-1}(g) \hookrightarrow \mathrm{sk}_0(g) \hookrightarrow \mathrm{sk}_1(g) \hookrightarrow \ldots \hookrightarrow \varinjlim \mathrm{sk}_*(g) = B$$

where each $\mathrm{sk}_{n-1}(g) \hookrightarrow \mathrm{sk}_n(g)$ is a semi-free extension of finitely many generators in degree n, together with map of simplicial commutative algebras

$$\begin{tikzcd} N^*D = N^*(\mathrm{sk}_{-1}(g)) \arrow[r, hook] \arrow[drr, "\phi^{(-1)}=f"'] & N^*(\mathrm{sk}_0(g)) \arrow[r, hook] \arrow[dr, "\phi^{(0)}"] & N^*(\mathrm{sk}_1(g)) \arrow[r, hook] \arrow[d, "\phi^{(1)}"] & \ldots \\ & & A & \end{tikzcd}$$

such that the map $\phi^{(n)} : N^*(sk_n(g)) \to A$ satisfies the following conditions:

1. It induces bijection $\phi^{(n)} : \pi_p(N^*(sk_n(g))) \to \pi_p(A)$ for all $p < n$.
2. The map $\phi^{(n)} : N^*(sk_n(g))_p \to A_p$ is surjective for all $p \leq n$
3. It induces a surjection $\phi^{(n)} : Z_n(N(N^*(sk_n(g)))) \to Z_n(N(A))$, and hence a surjection $\phi^{(n)} : \pi_n(N^*(sk_n(g))) \to \pi_n(A)$.

The pair $(\mathrm{sk}_0(g), \phi^{(0)})$ is easy to construct. Namely, choose generators $a_1^{(0)}, ..., a_{r_0}^{(0)} \in A_0$ of the commutative algebra A_0 over k. Define $\mathrm{sk}_0(g)$ to be the tensor product $\mathrm{sk}_0(g) = D \otimes k\langle x_1^{(0)}, ..., x_{r_0}^{(0)} \rangle$. Then $N^*(\mathrm{sk}_0(g))$ is a semi-free extension of $N^*(D)$, with additional non-degenerate generators $\eta(x_1^{(0)}), \ldots, \eta(x_{r_0}^{(0)})$, where η is the unit $\eta : \mathrm{sk}_0(g) \to NN^*(\mathrm{sk}_0(g))$ of the adjunction (11.0.1). Thus, to define the extension $\phi^{(0)} : N^*(\mathrm{sk}_0(g)) \to A$ of f, it suffices to specify the images of $\eta(x_i^{(0)})$, which we declare to be $a_i^{(0)}$. The map $\phi^{(0)}$ is clearly surjective in simplicial degree 0, and hence satisfies condition (2),(3) above for $n = 0$. Since condition (1) is vacuous in this case, we have constructed the desired pair $(\mathrm{sk}_0(g), \phi^{(0)})$.

Suppose that we have constructed the pair $(\mathrm{sk}_{n-1}(g), \phi^{(n-1)})$ for some $n \geq 1$. In order to construct the pair $(\mathrm{sk}_n(g), \phi^{(n)})$, we will add variables to $\mathrm{sk}_{n-1}(g)$ so as to ensure the conditions (1),(2),(3) above. For notational convenience, we will write $C := \mathrm{sk}_{n-1}(g)$ and $\psi := \phi^{(n-1)} : N^*(C) \to A$. In order to define a semi-free extension $C \hookrightarrow C'$ with (finitely many) generators $x_1^{(n)}, \dots, x_p^{(n)}$ in degree n, one needs to specify a set $z_1^{(n-1)}, \dots, z_p^{(n-1)}$ of elements in C_{n-1} so that we can define $d(x_i^{(n)}) = z_i^{(n-1)}$. These elements are required to satisfy $d(z_i^{(n-1)}) = 0$. In order to define an extension $\psi' : N^*(C') \to A$, which is equivalent to a map $\hat{\psi}' : C' \to N(A)$ of commutative DG algebras, one has to specify elements $c_1^{(n)}, \dots, c_p^{(n)}$ in $N(A)_n$ in order to define the map $\hat{\psi}'$ by $\hat{\psi}'(x_i^{(n)}) = c_i^{(n)}$, or equivalently, $\psi'(\eta(x_i^{(n)})) = c_i^{(n)}$. In other words, in extending (C, ψ) to (C', ψ'), one needs to specify elements $z_1^{(n-1)}, \dots, z_p^{(n-1)} \in C_{n-1}$ satisfying $d(z_i^{(n-1)}) = 0$, as well as elements $c_1^{(n)}, \dots, c_p^{(n)} \in N(A)_n \subset A_n$, such that $\psi(\eta(z_i^{(n-1)})) = d_0(c_i^{(n)})$.

We will now drop the superscripts that designate homological or simplicial degrees. In view of the above paragraph, we call a pair of tuples of elements $z_1, \dots, z_p \in C_{n-1}$ and $c_1, \dots, c_p \in N(A)_n \subset A_n$ satisfying $d(z_i) = 0$ and $\psi(\eta(z_i)) = d_0(c_i)$ an *extension pair*. In order to define the extension $(\mathrm{sk}_n(g), \phi^{(n)}) = (C', \psi')$, it suffices therefore to find

(a) an extension pair $(z_1, \dots, z_p)$ and $(c_1, \dots, c_p)$ such that the elements $[z_1], \dots, [z_p] \in H_{n-1}(C)$ lies in the kernel K of the (surjective) map $\hat{\psi} : H_{n-1}(C) \to H_{n-1}(N(A))$, and generates K over the commutative algebra $H_0(C)$;

(b) an extension pair $(z'_1, \dots, z'_q)$ and $(c'_1, \dots, c'_q)$ such that the elements $c'_1, \dots, c'_q \in A_n$ generates A_n over the image $\psi(N^*(C)_n) \subset A_n$ of ψ.

(c) and an extension pair $(z''_1, \dots, z''_r)$ and $(c''_1, \dots, c''_r)$ such that $z''_i = 0$, hence

$c_i'' \in Z_n(N(A))$, and that $c_1'', \ldots, c_r''$ generates $Z_n(N(A)) = \bigcap_{i=0}^{n} \ker [\, d_i : A_n \to A_{n-1} \,]$ as a module over A_n.

Indeed, these extension pairs determine an extension $(\mathrm{sk}_n(g), \phi^{(n)}) = (C', \psi')$. Since $C' = \mathrm{sk}_n(g)$ is a semi-free simplicial commutative algebra, we have $H_n(C') = H_n(N(N^*(C'))) = \pi_n(C')$. Thus, the extension pair (a) guarantees condition (1) of the extension $(\mathrm{sk}_n(g), \psi^{(n)})$. The extension pair (b) clearly guarantees conditions (2) for this extension. Moreover, in view of condition (2), the extension pair (c) is also sufficient to guarantee condition (3). This last implication is, in fact, the reason we impose the auxilliary condition (2) in the induction.

To construct the extension pair (a), we notice that since $H_0(C)$ is finitely generated and $H_n(C)$ is finitely generated as a module over $H_0(C)$. Therefore, the submodule $K \subset H_n(C)$ is also finitely generated over $H_0(C)$. Choose elements $z_1, \ldots, z_p \in C_{n-1}$ satisfying $d(z_i) = 0$ such that $[z_1], \ldots, [z_p]$ forms such a set of generators. By assumption, we have $\hat{\psi}([z_i]) = 0$, hence there exists elements $c_i \in N(A)_n$ such that $d(c_i) = \hat{\psi}(z_i)$. The pairs $(z_1, \ldots, z_p)$ and $(c_1, \ldots, c_p)$ then gives the desired extension pair (a).

To construct the extension pair (b), we start with any tuple $e_1, \ldots, e_q \in A_n$ that generates A_n over the subalgebra generated by degenerate elements $\bigcup_{i=0}^{n-1} s_i(A_{n-1})$. Now, by the isomorphism of vector spaces

$$N(A)_n := \bigcap_{i=1}^{n} \ker [\, d_i : A_n \to A_{n-1} \,] \xrightarrow{\cong} \frac{A_n}{\sum_{i=0}^{n-1} s_i(A_{n-1})} =: \overline{N}(A)_n$$

there exists, for each e_i, elements $f_{i,0}, \ldots, f_{i,n-1} \in A_n$ such that $e_i' = e_i + s_0(f_{i,0}) + \ldots + s_{n-1}(f_{i,n-1}) \in N(A)_n$ Then, $e_1', \ldots, e_q'$ still generate A_{n+1} over the subalgebra generated by the degenerate elements $s_i(A_n) \subset A_{n+1}$. By the induction hypothesis (2) of $(C, \psi) = (\mathrm{sk}_{n-1}(g), \phi^{(n-1)})$, this implies that $e_1', \ldots, e_q'$ generates A_{n+1}

over the image $\psi(N^*(C)_n) \subset A_n$ of ψ.

We would like to extend (a variant of) the tuple $(e'_1, \ldots, e'_q)$ to an extension pair. Thus, we want to write the elements $d_0(e'_i) \in Z_{n-1}(N(A))$ as the image under $\hat{\psi} : Z_{n-1}(C) \to Z_{n-1}(N(A))$. As a first approximation, we consider the map $N(\psi) : Z_{n-1}(N(N^*(C))) \to Z_{n-1}(N(A))$. This map is a surjection by inductive hypothesis (3), therefore one can write $d_0(e'_i) = \psi(u_i)$ for some $u_i \in Z_{n-1}(N(N^*(C)))$. While the elements u_i may not be of the form $\eta(z'_i)$ for some $z'_i \in Z_{n-1}(C)$, it is homologous to such an element. Indeed, since the unit map $\eta : C \to N(N^*(C))$ is a quasi-isomorphism, the element u_i can be written as $u_i = \eta(z'_i) + d(w_i)$ for some $w_i \in N(N^*(C))_n$. This shows that

$$\psi(\eta(z'_i)) = \psi(u_i) - \psi(d(w_i)) = d_0(e'_i) - d_0(\psi(w_i)) = d_0(e'_i - \psi(w_i))$$

Therefore, if we let $c'_i = e'_i - \psi(w_i) \in N(A)_n$, then the tuples $(z'_1, \ldots, z'_q)$ and $(c'_1, \ldots, c'_q)$ forms an extension pair. Moreover, since c'_i differs by e'_i from an element in $\psi(N(N^*(C))_n) \subset \psi(N^*(C)_n)$, it still generates A_n over $\psi(N^*(C)_n)$.

Finally, to construct the extension pair (c), notice that the submodule $Z_n(N(A)) \subset A_n$ is finitely generated as a module over A_n. Choose a set of generators $c''_1, \ldots, c''_r \in Z_n(N(A))$. The pair $(0, \ldots, 0)$ and $(c''_1, \ldots, c''_r)$ then forms a desired extension pair. This finishes the inductive step, and hence the proof of the lemma. $\square$

Remark 11.1.5. In Lemma 11.1.4, the assumption that D is semi-free is used only to guarantee that the unit map $\mathrm{sk}_n(g) \to NN^*(\mathrm{sk}_n(g))$ is a quasi-isomorphism for each $n \geq -1$. Thus, one can weaken this assumption by only requiring that D be adapted under the functor N^*.

We summarize our arguments in the following

Proof of Proposition 11.1.1. We have seen that $(2) \Rightarrow (1)$ is trivial, while $(2) \Rightarrow (3)$ follows from the Quillen equivalence (11.0.1). To show the implication $(1) \Rightarrow (2)$, simply apply Lemma 11.1.3 to the map $k \hookrightarrow N(A)$, which gives a weak equivalence $\tilde{B} \xrightarrow{\sim} N(A)$, such that $\tilde{B}$ is semi-free with finitely many generators in each degree. To show the implication $(3) \Rightarrow (2)$, we apply Lemma 11.1.4 to the map $D = N^*(k) = k \to A$, which gives a weak equivalence $N^*(B) \xrightarrow{\sim} A$ where B is semi-free with finitely many generators in each degree. In view of the Quillen equivalence (11.0.1), this again correspond to a weak equivalence $B \xrightarrow{\sim} N(A)$. $\square$

11.2 Reedy diagrams

In this subsection, we recall some basic notions about Reedy categories, mostly following [77, Chapter 15].

Definition 11.2.1. A *Reedy category* is a small category I, together with two subcategories $\overrightarrow{I}$ (the *direct subcategory*) and $\overleftarrow{I}$ (the *inverse subcategory*), both containing all the objects of I, together with a degree function $\deg : \mathrm{Ob(I)} \to \mathbb{N}$ such that

1. Every non-identity morphism of $\overrightarrow{I}$ raises degree.
2. Every non-identity morphism of $\overleftarrow{I}$ lowers degree.
3. Every morphism g in I can be factorized uniquely as $g = \overrightarrow{g}\overleftarrow{g}$ where $\overrightarrow{g} \in \overrightarrow{I}$ and $\overleftarrow{g} \in \overleftarrow{I}$.

For a Reedy category I, denote by $I^{(n)}$ the full subcategory of I consisting of objects of degree n. Similarly, define $I^{(\leq n)}$, $\overrightarrow{I}^{(\leq n)}$, etc, to be the obvious full subcategories corresponding to objects with degrees specified by the subscripts. Given a diagram $X : I \to \mathcal{M}$ to any complete and cocomplete category $\mathcal{M}$, one can consider the restrictions $X^{(\leq n)} = X|_{I^{(\leq n)}} : I^{(\leq n)} \to \mathcal{M}$ of X to each of the full subcategories $I^{(\leq n)}$, and investigate how $X^{(\leq n)}$ extends $X^{(\leq n-1)}$. The extremal cases is when $X^{(\leq n)}$ is a left or right Kan extension of $X^{(\leq n-1)}$. In the case of left Kan extension, the value at each object $\alpha \in I^{(n)}$ of degree n is given by the following *latching object*

$$L_\alpha(X) = \operatorname*{colim}_{\substack{\beta \to \alpha \\ \deg(\beta) \leq n-1}} X_\beta$$

which depends only on the restriction $X_{(\leq n-1)}$. Similarly, in the case of right Kan extension, the value at each object $\alpha \in I_{(n)}$ of degree n is given by the following *matching object*

$$M_\alpha(X) = \lim_{\substack{\alpha \to \beta \\ \deg \beta \leq n-1}} X_\beta$$

In general, there is no reason to expect that $X_{(\leq n)}$ is either a left or a right Kan extension of $X_{(\leq n-1)}$. Thus, X_α will be somewhere in between. Namely, there are map

$$L_\alpha(X) \to X_\alpha \to M_\alpha(X) \tag{11.2.2}$$

that factorizes the canonical map $L_\alpha(X) \to M_\alpha(X)$. In fact, choosing such a factorization (11.2.2) for each object α of degree n uniquely determines an extension of $X_{(\leq n-1)}$ to $X_{(\leq n)}$. (see [77, Theorem 15.2.1])

There is an alternative description of the latching and matching objects that will be useful for us later. Consider the overcategory $\overrightarrow{I} \downarrow \alpha$. This overcategory contains a final object id_α. Take the full subcategory $\partial(\overrightarrow{I} \downarrow \alpha) \subset \overrightarrow{I} \downarrow \alpha$, called the *latching category*, containing all the objects except the final object. In other words, $\partial(\overrightarrow{I} \downarrow \alpha)$ is isomorphic to the category $\overrightarrow{I}^{(\leq n-1)} \downarrow \alpha$. Therefore, one has an inclusion functor

$$\partial(\overrightarrow{I} \downarrow \alpha) = (\overrightarrow{I}^{(\leq n-1)} \downarrow \alpha) \rightarrow (I^{(\leq n-1)} \downarrow \alpha)$$

It turns out that this inclusion functor is right cofinal [77, Proposition 15.2.8, Definition 14.2.1], and hence induces an isomorphism

$$L_\alpha(X) = \operatorname*{colim}_{\substack{(\beta\to\alpha)\in I \\ \deg(\beta)\leq n-1}} X_\beta = \operatorname*{colim}_{\substack{(\beta\to\alpha)\in \overrightarrow{I} \\ \deg(\beta)\leq n-1}} X_\beta \tag{11.2.3}$$

Dually, one has

$$M_\alpha(X) = \lim_{\substack{(\alpha\to\beta)\in I \\ \deg(\beta)\leq n-1}} X_\beta = \lim_{\substack{(\alpha\to\beta)\in \overleftarrow{I} \\ \deg(\beta)\leq n-1}} X_\beta \tag{11.2.4}$$

Suppose we are given a map $f : X \to Y$ of I-diagrams in $\mathcal{M}$. Since any I-diagram X is determined inductively by the latching and matching maps (11.2.2), it is natural to investigate the effect of f on these maps. In other words, consider the commutative diagram

$$\begin{tikzcd}
L_\alpha X \arrow[r] \arrow[d] & X_\alpha \arrow[r] \arrow[d] & M_\alpha X \arrow[d] \\
L_\alpha Y \arrow[r] & Y_\alpha \arrow[r] & M_\alpha Y
\end{tikzcd}$$

which is equivalent to giving

1. the *relative latching map* $f : X_\alpha \coprod_{L_\alpha X} L_\alpha Y \to Y_\alpha$; and
2. the *relative latching map* $f : X_\alpha \to Y_\alpha \times_{M_\alpha Y} M_\alpha X$

Again, one can show that a map $f : X \to Y$ of I-diagrams in $\mathcal{M}$ is determined inductively by the relative latching and matching maps.

Now, if $\mathcal{M}$ is a model category, then these considerations allow one to give a model structure on the category $\mathcal{M}^I$ of I shaped diagrams in $\mathcal{M}$. We say that a map $f : X \to Y$ in $\mathcal{M}^I$ is a *Reedy cofibration* if all the relative latching maps are cofibrations in $\mathcal{M}$, Dually, we say that a map $f : X \to Y$ in $\mathcal{M}^I$ is a *Reedy fibration* if all the relative matching maps are fibrations in $\mathcal{M}$. Then we have the following Theorem. For a convenient reference, see [77, Theorem 15.3.4].

Theorem 11.2.5. *The category $\mathcal{M}^I$ has a model structure, called the Reedy model structure, in which weak equivalences are pointwise weak equivalences; cofibrations are Reedy cofibrations; and fibrations are Reedy fibrations.*

In the Reedy model category $\mathcal{M}^I$, an object X is cofibrant if and only if each of the latching maps $L_\alpha(X) \to X_\alpha$ is a cofibration in $\mathcal{M}$. It is shown in [77, Corollary 15.3.12] that if X is Reedy cofibrant, then it is objectwise cofibrant, *i.e.,* each X_α is a cofibrant object in $\mathcal{M}$. In the rest of this subsection, we will give an alternative proof of this fact by exhibiting each object X_α as a composition of pushouts of coproducts of latching morphisms (see (11.2.12) and (11.2.13)). We will use the assumption that X be Reedy cofibrant only at the very last step. This allows us to apply the results here to the next subsection, specifically Lemma 11.3.7. Thus, we will assume throughout the rest of this subsection that $X : I \to \mathcal{M}$ is any I-shaped diagram in $\mathcal{M}$, not necessarily Reedy cofibrant.

First, consider the diagram

$$\begin{tikzcd}
\coprod_{\alpha\in \mathrm{Ob}(\mathrm{I}^{(\mathrm{n})})} \left[\operatorname*{colim}_{\substack{(\beta\to\alpha)\in\overrightarrow{I} \\ \deg(\beta)\le n-1}} X_\beta \right] \arrow[r, "F"] \arrow[d, "G"] & \coprod_{\alpha\in \mathrm{Ob}(\mathrm{I}^{(\mathrm{n})})} X_\alpha \arrow[d, "G'"] \\
\operatorname*{colim}_{\beta\in(\overrightarrow{I}^{(\le n-1)})} X_\beta \arrow[r, "F'"] & \operatorname*{colim}_{\gamma\in(\overrightarrow{I}^{(\le n)})} X_\gamma
\end{tikzcd} \tag{11.2.6}$$

To specify the maps F and G, consider the copy X_β corresponding to the map $g : \beta \to \alpha$. Then F sends this copy of X_β to X_α via the map $X(g)$, while

G sends it to X_β via the identity map. The maps F' and G' are induced by the obvious inclusions. Unravelling the definitions, one sees immediately that this diagram commutes.

Lemma 11.2.7. *The diagram* (11.2.6) *is a pushout diagram.*

Proof. Specifying a commutative diagram

$$\begin{tikzcd}
\coprod_{\alpha\in\mathrm{Ob}(\mathrm{I}^{(\mathrm{n})})}\left[\operatorname*{colim}_{\substack{(\beta\to\alpha)\in\overrightarrow{I}\\ \deg(\beta)\le n-1}} X_\beta\right] \arrow[r, "F"] \arrow[d, "G"] & \coprod_{\alpha\in\mathrm{Ob}(\mathrm{I}^{(\mathrm{n})})} X_\alpha \arrow[d, "G''"] \\
\operatorname*{colim}_{\beta\in(\overrightarrow{I}^{(\le n-1)})} X_\beta \arrow[r, "F''"] & Z
\end{tikzcd} \tag{11.2.8}$$

is equivalent to giving a map $f_\alpha : X_\alpha \to Z$ for each object $\alpha \in I^{(n)}$ of degree n, as well as a map $f_\beta : X_\beta \to Z$ for each object $\beta \in I^{(\le n-1)}$ of degree $\le n-1$, subject to the conditions that the following two collections of diagrams commute

$$\begin{tikzcd}
X(\beta_1) \arrow[rr, "X(g)"] \arrow[dr, "f_{\beta_1}"'] & & X(\beta_2) \arrow[dl, "f_{\beta_2}"] \\
& Z &
\end{tikzcd}
\qquad
\begin{tikzcd}
X(\beta) \arrow[rr, "X(h)"] \arrow[dr, "f_{\beta}"'] & & X(\alpha) \arrow[dl, "f_{\alpha}"] \\
& Z &
\end{tikzcd} \tag{11.2.9}$$

where $g : \beta_1 \to \beta_2$ is a map in $\overrightarrow{I}$ between two objects of degree $\le n-1$, and $h : \beta \to \alpha$ is a map in $\overrightarrow{I}$ from an object of degree $\le n-1$ to an object of degree n.

Indeed, the map G'' in (11.2.8) induces the collection of maps $f_\alpha : X_\alpha \to Z$ for $\alpha \in I^{(n)}$ of degree n. The map F'' in (11.2.8) induces the collection of maps $f_\beta : X_\beta \to Z$ for $\beta \in I^{(\le n-1)}$ of degree $\le n-1$ making each diagrams to the left of (11.2.9) commute. Then, the commutativity of (11.2.8) is equivalent to saying that each diagrams to the right of (11.2.9) commutes.

It is then clear that given this data is equivalent to giving a map $f : \operatorname*{colim}_{\gamma\in(\overrightarrow{I}^{(\leq n)})} X_\gamma \to Z$. This is precisely the statement we wish to prove. $\square$

Now, notice that the colimit appearing in the square bracket in the top left entry of (11.2.6) is simply the latching object $L_\alpha(X)$, while the map F in (11.2.6) is simply the coproduct of the latching morphisms $L_\alpha(X) \to X_\alpha$. In other words, we have a pushout diagram

$$\begin{tikzcd} \coprod_{\alpha\in\mathrm{Ob(I^{(n)})}} L_\alpha(X) \arrow[r] \arrow[d] & \coprod_{\alpha\in\mathrm{Ob(I^{(n)})}} X_\alpha \arrow[d] \\ \operatorname*{colim}_{\gamma\in(\overrightarrow{I}^{(\leq n-1)})} X_\gamma \arrow[r] & \operatorname*{colim}_{\gamma\in(\overrightarrow{I}^{(\leq n)})} X_\gamma \end{tikzcd} \tag{11.2.10}$$

We will apply this to construct each object X_α as a successive pushout of latching morphisms. To this end, consider the overcategory $J_\alpha := (\overrightarrow{I} \downarrow \alpha)$. Notice that J_α is a Reedy category, with $\overrightarrow{J_\alpha} = J_\alpha$ and $\overleftarrow{J_\alpha}$ consisting only of identity maps, while the degree map is inherited from that of I. The diagram $X : I \to \mathcal{M}$ induces a J-shaped diagram $Y_\alpha : J_\alpha \to \mathcal{M}$ defined as the composition

$$Y_\alpha \,:\, J_\alpha = (\overrightarrow{I} \downarrow \alpha) \xrightarrow{\pi} \overrightarrow{I} \hookrightarrow I \xrightarrow{X} \mathcal{M}$$

In other words, Y_α is given by $Y_\alpha(\beta \to \alpha) := X_\beta$. It is clear that the latching category $\partial(\overrightarrow{Y_\alpha} \downarrow (\beta \to \alpha))$ of Y_α at any object $\beta \to \alpha$ is isomorphic to the latching category $\partial(\overrightarrow{X} \downarrow \beta)$ of X at β. Therefore, the latching objects as well as latching maps of Y_α coincide with those of X:

$$\big[\, L_{(\beta\to\alpha)}(Y_\alpha) \to Y_\alpha(\beta \to \alpha) \big] \cong \big[\, L_\beta(X) \to X_\beta \big] \tag{11.2.11}$$

Therefore, the pushout diagram (11.2.10) corresponding to the Reedy diagram $Y : J \to \mathcal{M}$ takes the form

$$\begin{tikzcd} \coprod\limits_{\substack{\beta\in \mathrm{Ob}(\mathrm{I}^{(\mathrm{r})}) \\ \beta\to\alpha}} L_\beta(X) \arrow[r] \arrow[d] & \coprod\limits_{\substack{\beta\in \mathrm{Ob}(\mathrm{I}^{(\mathrm{r})}) \\ \beta\to\alpha}} X_\beta \arrow[d] \\ \operatorname*{colim}\limits_{\gamma\in J_\alpha^{(\leq r-1)}} Y_\alpha(\gamma) \arrow[r] & \operatorname*{colim}\limits_{\gamma\in J_\alpha^{(\leq r)}} Y_\alpha(\gamma) \end{tikzcd} \tag{11.2.12}$$

Now, notice that $J_\alpha^{(\leq n-1)}$ is precisely the latching category $\partial(\overrightarrow{I} \downarrow \alpha)$ of X at α. Therefore, we have a sequence of maps

$$\emptyset = \operatorname*{colim}_{\gamma\in J_\alpha^{(\leq -1)}} Y_\alpha(\gamma) \to \operatorname*{colim}_{\gamma\in J_\alpha^{(\leq 0)}} Y_\alpha(\gamma) \to \operatorname*{colim}_{\gamma\in J_\alpha^{(\leq 1)}} Y_\alpha(\gamma) \to \dots \to \operatorname*{colim}_{\gamma\in J_\alpha^{(\leq n-1)}} Y_\alpha(\gamma) = L_\alpha(X) \tag{11.2.13}$$

each of which is a pushout of coproducts of latching morphisms, as in (11.2.12). This allows one to prove the following

Proposition 11.2.14. *If $X : I \to \mathcal{M}$ is a Reedy cofibrant diagram, then X is objectwise cofibrant.*

Proof. If X is Reedy cofibrant, then each of the maps in the sequence (11.2.13) is a pushout (11.2.12) of a coproduct of cofibrations, and hence a cofibration. As a consequence, the latching object $L_\alpha(X)$ is cofibrant. Since $L_\alpha(X) \to X_\alpha$ is also assumed to be a cofibration, the same holds for the object X_α. $\square$

11.3 Smooth extensions

For any $[A] \in \mathrm{Ho}(\mathtt{sCommAlg}_k)$, we can define an augmentation of $[A]$ to be either a map $[A] \to [k]$ in $\mathrm{Ho}(\mathtt{sCommAlg}_k)$, or a map $A \to k$ in $\mathtt{sCommAlg}_k$, or a map $\pi_0(A) \to k$ in $\mathtt{CommAlg}_k$ (which are clearly equivalent). We will denote an augmentation as ε_x, thinking of it as a k-point on the affine scheme $\mathrm{Spec}(\pi_0(A))$.

Any augmentation ε_x allows one to linearize A. Namely, let $\mathfrak{m}_x$ be the kernel of the map $\varepsilon_x : A \to k$, then we define $\mathcal{Q}_x(A) := \mathfrak{m}_x / \mathfrak{m}_x^2$. This gives the linearization functor from the category of augmented simplicial commutative algebra to the category of simplicial k-modules

$$\mathcal{Q} : \mathtt{sCommAlg}_{k/k} \to \mathtt{sMod}_k .$$

The functor $\mathcal{Q}$ has a right adjoint, given by assigning the simplicial commutative algebra $k \oplus M$ to a simplicial k-module M. This forms a Quillen adjunction, and allows one to define the derived functor

$$\boldsymbol{L}\mathcal{Q} : \mathrm{Ho}(\mathtt{sCommAlg}_{k/k}) \to \mathrm{Ho}(\mathtt{sMod}_k) ,$$

which is the classical cotangent complex $\mathbb{L}^{A/k}$ (see [6, 137]). Given a map $f : [A] \to [B]$ in $\mathrm{Ho}(\mathtt{sCommAlg}_k)$, any augmentation $\varepsilon_x : [B] \to [k]$ of $[B]$ determines an augmentation $\varepsilon_{f^*(x)} : [A] \xrightarrow{f} [B] \xrightarrow{\varepsilon_x} [k]$ of $[A]$. The map f is said to be *étale* at x if the induced map

$$f : \boldsymbol{L}\mathcal{Q}_{f^*(x)}(A) \to \boldsymbol{L}\mathcal{Q}_x(B)$$

is an isomorphism in $\mathrm{Ho}(\mathtt{sMod}_k)$.

Let $\bar{k}$ be the algebraic closure of k. Then a map $f : [A] \to [B]$ in $\mathrm{Ho}(\mathtt{sCommAlg}_k)$ is called *étale* if its base change $\tilde{f} : [A_{\bar{k}}] \to [B_{\bar{k}}]$ to the algebraic closure $\bar{k}$ is étale for every augmentation $\varepsilon_x : [B_{\bar{k}}] \to [\bar{k}]$. The proposition below roughly says that a simplicial commutative algebra of quasi-finite type is determined by its underlying scheme as well as its derived cotangent spaces. When stated in terms of differential graded commutative algebras, this result follows either from [96, Proposition 1.3] or [20, Corollary 2.7]. The following version for simplicial commutative algebras then follows from Theorem 10.3.2.

Proposition 11.3.1. *Let $f : [A] \to [B]$ be a morphism in Ho$(\mathtt{sCommAlg}_k)$ between two objects of quasi-finite type. Suppose f is étale and induces an isomorphism in $\pi_0(f) : \pi_0(A) \xrightarrow{\cong} \pi_0(B)$. Then f is an isomorphism in Ho$(\mathtt{sCommAlg}_k)$.*

Homotopical arguments often require imposing the cofibrancy condition on objects and morphisms between simplicial commutative algebras. As observed in [96], the above proposition allows one to replace the cofibrancy condition by certain smoothness conditions. To this end, we introduce the following replacement of cofibrations.

Definition 11.3.2. We say that a map $f : A \to B$ of simplicial commutative algebras is a *smooth extension* if it can be written as an (infinite) composition

$$A = \mathrm{sk}_{-1}(f) \to \mathrm{sk}_0(f) \to \mathrm{sk}_1(f) \to \ldots \to \varinjlim \mathrm{sk}_*(f) = B$$

such that each map $\mathrm{sk}_{n-1}(f) \to \mathrm{sk}_n(f)$ is a pushout of the form

$$\begin{tikzcd} \partial\Delta[n] \otimes C_n \arrow[r] \arrow[d] & \Delta[n] \otimes C_n \arrow[d] \\ \mathrm{sk}_{n-1}(f) \arrow[r] & \mathrm{sk}_n(f) \end{tikzcd} \tag{11.3.3}$$

where C_n is some smooth commutative algebra in $\mathtt{CommAlg}_k$. Then, we say that a simplicial commutative algebra A is *smooth* if the structure map $k \to A$ is a smooth extension.

Notice that, when the commutative algebras C_n appearing in this definition are polynomial algebras $C_n = k[x_{n,1}, \ldots, x_{n,r_n}]$, then the map $f : A \to B$ is a semi-free extension of finitely many generators in each simplicial degree. Thus, smooth extensions generalizes quasi-finite semi-free extensions.

The class of smooth extension is preserved under compositions and pushouts. Moreover, if we focus on cotangent complexes, then the following proposition shows that smooth objects and smooth extensions between objects share the same crucial properties as cofibrant objects and cofibrations between objects.

Proposition 11.3.4. (1) *Suppose that A is a smooth simplicial commutative algebra, then for any augmentation $\varepsilon_x : A_{\bar{k}} \to \bar{k}$, the canonical map $\boldsymbol{L}\mathcal{Q}_x(A_{\bar{k}}) \to \mathcal{Q}_x(A_{\bar{k}})$ is an isomorphism in Ho$(\mathtt{sMod}_{\bar{k}})$.*
(2) *Suppose that $f : A \to B$ is a smooth extension of simplicial commutative algebras, then for any augmentation $\varepsilon_x : B_{\bar{k}} \to \bar{k}$, the induced map $\mathcal{Q}_x f : \mathcal{Q}_{f^*(x)}(A_{\bar{k}}) \to \mathcal{Q}_x(B_{\bar{k}})$ on the linearizations is a cofibration in $\mathtt{sMod}_{\bar{k}}$.*

Proof. To show (1), we recall a classical fact (see [6, 137]) that any smooth algebra C is adapted (see Definition 9.2.1) under the linearization functor $\mathcal{Q}_x$ of any augmentation $\varepsilon_x : C_{\bar{k}} \to \bar{k}$. Now, if A is a smooth simplicial commutative algebra, then in each simplicial degree n, the commutative algebra A_n is smooth, and hence adapted under $\mathcal{Q}_x$, for any augmentation ε_x of A. One can then apply Theorem 9.2.2 to conclude part (1). Notice that the category $\mathtt{CommAlg}_{k/k}$ is equivalent to the category of non-unital commutative algebras over k, and is therefore an algebraic category. Thus, the linearization functor $\mathcal{Q} : \mathtt{sCommAlg}_{k/k} \to \mathtt{sMod}_k$ is the simplicial prolongation of a left adjoint functor $\mathcal{Q} : \mathtt{CommAlg}_{k/k} \to \mathtt{Mod}_k$ between algebraic categories. Therefore, we are indeed in the setting of Theorem 9.2.2.

To show (2), write $f : A \to B$ as an infinite composition as in Definition 11.3.2. The augmentation $\varepsilon_x : B_{\bar{k}} \to \bar{k}$ then induces an augmentation, still denoted as ε_x, of each skeleton $\mathrm{sk}_n(f)$. This further induces a compatible system of augmentations of each of the algebras appearing in (11.3.3), which allows us to apply the linearization functor $\mathcal{Q} : \mathtt{sCommAlg}_{k/k} \to \mathtt{sMod}_k$ to each of the diagrams (11.3.3). Since the linearization functor $\mathcal{Q}$ is a left adjoint, it commutes with colimits. Moreover, since colimits in the over-category $\mathtt{sCommAlg}_{k/k}$ can be computed as a colimit in the category $\mathtt{sCommAlg}_k$, the diagram

$$\begin{tikzcd} \mathcal{Q}_x\left(\,\partial\Delta[n]\otimes C_n\,\right) \arrow[r] \arrow[d] & \mathcal{Q}_x\left(\,\Delta[n]\otimes C_n\,\right) \arrow[d] \\ \mathcal{Q}_x\left(\,\mathrm{sk}_{n-1}(f)\,\right) \arrow[r] & \mathcal{Q}_x\left(\,\mathrm{sk}_n(f)\,\right) \end{tikzcd} \tag{11.3.5}$$

is still a pushout diagram in the category $\mathtt{sMod}_k$ of simplicial k-modules. As the simplicial prolongation of a left adjoint functor, $\mathcal{Q}_x$ commutes with copowering $K\otimes -$ by simplicial sets K. Therefore, the map in the top row of (11.3.5) is isomorphic to $\partial\Delta[n]\otimes\mathcal{Q}_x(C_n)\ \rightarrow\ \Delta[n]\otimes\mathcal{Q}_x(C_n)$, and is therefore a cofibration in $\mathtt{sMod}_k$. This shows that the maps $\mathcal{Q}_x\left(\,\mathrm{sk}_{n-1}(f)\,\right)\ \rightarrow\ \mathcal{Q}_x\left(\,\mathrm{sk}_{n-1}(f)\,\right)$ are cofibrations, and hence so is their infinite composition $\mathcal{Q}_x(f):\mathcal{Q}_x(A)\ \rightarrow\ \mathcal{Q}_x(B)$. $\square$

Definition 11.3.6. Given a Reedy category I, a diagram $A:I\ \rightarrow\ \mathtt{sCommAlg}_k$ is called *Reedy smooth* if for each $\alpha\in I$, the latching map $f_\alpha\ :\ L_\alpha(A)\ \rightarrow\ A(\alpha)$ is a smooth extension.

Recall from Proposition 11.2.14 that if $F\ :\ I\ \rightarrow\ \mathcal{M}$ is a Reedy cofibrant diagram in any model category $\mathcal{M}$, then F is objectwise cofibrant. A similar statement is true for Reedy smooth diagrams:

Lemma 11.3.7. *Let $A:I\ \rightarrow\ \mathtt{sCommAlg}_k$ be a Reedy smooth diagram, then each $A(\alpha)$ is a filtered colimit of smooth simplicial commutative algebras. Moreover, if the Reedy category I is locally finite, meaning that there are only finitely many objects in I in each degree, and all the morphism sets in I are finite, then $A(\alpha)$ is in fact a smooth simplicial commutative algebra.*

Proof. Applying to the Reedy diagram A, the maps in (11.2.13) expresses the structure map $k \to L_\alpha(A)$ as a composition of maps $\operatorname*{colim}_{\gamma\in J_\alpha^{(\leq r-1)}} Y_\alpha(\gamma) \to \operatorname*{colim}_{\gamma\in J_\alpha^{(\leq r)}} Y_\alpha(\gamma)$, each of which is, by (11.2.12), a pushout of coproducts of latching morphisms, which are assumed to be smooth extensions.

If I is locally finite, then each of the coproducts appearing in the first row of (11.2.12) is a finite coproduct, hence both rows of (11.2.12) are smooth extensions. The composition (11.2.13) is therefore a smooth extension, showing that $L_\alpha(A)$ is smooth. Since $L_\alpha(A) \to A(\alpha)$ is also assumed to be a smooth extension, the same holds for $A(\alpha)$.

We may think of each of the colimit $\operatorname*{colim}_{\gamma\in J_\alpha^{(\leq r)}} Y_\alpha(\gamma)$ as the r-th skeleton of the latching object $L_\alpha(A)$, and the pushout (11.2.12) as attaching cells indexed by the maps $\beta \to \alpha$ for $\deg(\beta) = r$. In general, without the local finiteness assumption on I, there may be infinitely many cells in each degree. However, we may choose finite subcollections of cells such that each of the cell attaching maps $L_\beta(A) \to \operatorname*{colim}_{\gamma\in J_\alpha^{(\leq r-1)}} Y_\alpha(\gamma)$ lies in those cells. Each cell lies in such a finite subcollection. This allows one to express $L_\alpha(A)$, and similarly $A(\alpha)$, as a filtered colimit of smooth simplicial commutative algebras. $\square$

With this lemma, one can apply Proposition 11.3.4 to deduce the following

Lemma 11.3.8. *Suppose that $A : I \to \mathtt{sCommAlg}_k$ is a Reedy smooth diagram, then for each $\alpha \in I$ and each $\bar{k}$-valued augmentation $\varepsilon_x : A(\alpha)_{\bar{k}} \to \bar{k}$, we have*

(1) *The canonical map $\boldsymbol{L}\mathcal{Q}_x(A(\alpha)_{\bar{k}}) \to \mathcal{Q}_x(A(\alpha)_{\bar{k}})$ is an isomorphism in $Ho(\mathtt{sMod}_{\bar{k}})$.*

(2) *The latching map $f_\alpha : L_\alpha(A) \to A(\alpha)$ induces a cofibration on the linearization $\mathcal{Q}_x f_\alpha : \mathcal{Q}_{f^*(x)}(L_\alpha(A)_{\bar{k}}) \to \mathcal{Q}_x A(\alpha)_{\bar{k}}$ in $\mathtt{sMod}_{\bar{k}}$.*

We will be primarily concerned with the case when the Reedy category I is the simplicial category Δ. In this case, one can tensor any diagram $A : \Delta \to \mathtt{sCommAlg}_k$ with a simplicial set $X : \Delta^{\mathrm{op}} \to \mathtt{Set}$ to obtain $X \otimes_\Delta A \in \mathtt{sCommAlg}_k$. This gives a functor

$$- \otimes_\Delta A : \mathtt{sSet} \to \mathtt{sCommAlg}_k \tag{11.3.9}$$

which is a left Kan extension of the functor $A : \Delta \to \mathtt{sCommAlg}_k$ along the Yoneda embedding $j : \Delta \to \mathtt{sSet}$.

The latching maps for A can be written in terms of these tensor products. In fact, by [77, Proposition 16.3.8], the latching map for A at $[n] \in \Delta$ can be identified with

$$L_\alpha(A) \cong \partial\Delta[n] \otimes_\Delta A \longrightarrow \Delta[n] \otimes_\Delta A \cong A(\alpha) \tag{11.3.10}$$

For each fixed X, the functor $X \otimes_\Delta - : (\mathtt{sCommAlg}_k)^\Delta \to \mathtt{sCommAlg}_k$ has a total left derived functor. The corresponding derived tensor product $X \otimes^{\boldsymbol{L}}_\Delta A \in \mathrm{Ho}(\mathtt{sCommAlg}_k)$ is represented by the tensor prodcut $X \otimes_\Delta Q$ for any Reedy cofibrant replacement Q of A.

Remark 11.3.11. The reader is cautioned that, despite the notation, the derived tensor product $X \otimes^{\boldsymbol{L}}_\Delta A$ is *not* homotopy invariant in X. In other words, if $X \xrightarrow{\sim} Y$ is a homotopy equivalence of simplicial sets, then the induced map $X \otimes^{\boldsymbol{L}}_\Delta A \to Y \otimes^{\boldsymbol{L}}_\Delta A$ may not be a weak equivalence in $\mathtt{sCommAlg}_k$. In fact, since $\mathtt{sCommAlg}_k$ is a simplicial model category, the tensor product functor $- \otimes_\Delta -$ is a Quillen bifunctor

$$- \otimes_\Delta - : (\mathtt{sSet})^{\Delta^{\mathrm{op}}} \times (\mathtt{sCommAlg}_k)^\Delta \to \mathtt{sCommAlg}_k$$

when both $(\mathtt{sSet})^{\Delta^{\mathrm{op}}}$ and $(\mathtt{sCommAlg}_k)^\Delta$ are endowed with the Reedy model structures. The derived tensor product $X \otimes^{\boldsymbol{L}}_\Delta A$ that we consider here can be regarded as the total left derived functor of this Quillen bifunctor, restricted to the case when $X \in \mathtt{s}(\mathtt{Set}) \subset \mathtt{s}(\mathtt{sSet}) = (\mathtt{sSet})^{\Delta^{\mathrm{op}}}$. Notice that, a map $f : X \to Y$ in $\mathtt{s}(\mathtt{Set})$ is a weak equivalence in the Reedy model category $(\mathtt{sSet})^{\Delta^{\mathrm{op}}}$ if and only if f is an isomorphism of simplicial sets. Therefore, the derived tensor product $X \otimes_{\Delta}^{\boldsymbol{L}} A$ may not be homotopy invariant in X even though it arises from a Quillen bifunctor.

Recall that a simplicial set X is *finite* if it has finitely many nondegenerate elements. The following lemma asserts that tensoring over a finite simplicial set preserves quasi-finiteness.

Lemma 11.3.12. *Let X be a finite simplicial set. Suppose that $A : \Delta \to \mathtt{sCommAlg}_k$, $[n] \mapsto \{A_m^n\}_{m\geq 0}$ is a cosimplicial simplicial commutative algebra such that each $A_m^n \in \mathtt{CommAlg}_k$ is finitely generated, then both the tensor product $X \otimes_{\Delta} A$ and the derived tensor product $X \otimes_{\Delta}^{\boldsymbol{L}} A$ are of quasi-finite type.*

Proof. We will show inductively on the dimension (*i.e.*, the highest degree of non-degenerate elements) of the simplicial set X that $X \otimes_{\Delta} A$ is degreewise finitely generated, and hence of quasi-finite type by the condition (3) of Proposition 11.1.1. The case for dimension 0 is easy. Suppose the statements are proved for dimension $< n$, and suppose that X is of dimension n. Let E_r be the set of non-degenerate elements of X of degree r. Then X can be regarded as a finite composition

$$\emptyset = \mathrm{sk}_{-1}(X) \hookrightarrow \mathrm{sk}_0(X) \hookrightarrow \mathrm{sk}_1(X) \hookrightarrow \dots \hookrightarrow \mathrm{sk}_{n-1}(X) \hookrightarrow \mathrm{sk}_n(X) = X$$

and each map $\mathrm{sk}_{r-1}(X) \hookrightarrow \mathrm{sk}_r(X)$ is a pushout

$$\begin{tikzcd} \coprod_{E_r} \partial\Delta^r \arrow[r] \arrow[d] & \coprod_{E_r} \Delta^r \arrow[d] \\ \mathrm{sk}_{r-1}(X) \arrow[r] & \mathrm{sk}_r(X) \end{tikzcd}$$

As a left Kan extension, $-\otimes_\Delta A$ commutes with colimits. Therefore, we have a pushout

$$\begin{tikzcd}
(\partial\Delta^r \otimes_\Delta A)^{\otimes E_r} \arrow[r] \arrow[d] & (\Delta^r \otimes_\Delta A)^{\otimes E_r} \arrow[d] \\
\mathrm{sk}_{r-1}(X) \otimes_\Delta A \arrow[r] & \mathrm{sk}_r(X) \otimes_\Delta A
\end{tikzcd}$$

In particular, putting $r = n$, both entries in the first column of this commutative diagram are degreewise finitely generated by the induction hypothesis. Moreover, the top right corner is given by a finite tensor product of $\Delta^n \otimes_\Delta A = A^n_* \in \mathtt{sCommAlg}_k$, and is therefore degreewise finitely generated. This shows that the pushout $\mathrm{sk}_r(X) \otimes_\Delta A$ is also degreewise finitely generated.

To prove that the derived tensor product $X \otimes_\Delta A$ is of quasi-finite type, we again proceed inductively on the dimension of X. Thus, replace A by a Reedy cofibrant diagram Q, and again express $X \otimes_\Delta Q$ as a similar composition of pushouts

$$\begin{tikzcd}
(\partial\Delta^r \otimes_\Delta Q)^{\otimes E_r} \arrow[r] \arrow[d] & (\Delta^r \otimes_\Delta Q)^{\otimes E_r} \arrow[d] \\
\mathrm{sk}_{r-1}(X) \otimes_\Delta Q \arrow[r] & \mathrm{sk}_r(X) \otimes_\Delta Q
\end{tikzcd}$$

Putting $r = n$, both entries in the first column of this commutative diagram are of quasi-finite type by the induction hypothesis. Moreover, the top right corner is given by a finite tensor product of $\Delta^n \otimes_\Delta Q = Q^n_* \simeq A^n_* \in \mathtt{sCommAlg}_k$, and is therefore of quasi-finite type.

Now, notice that the first row of this diagram is a cofibration in $\mathtt{sCommAlg}_k$ by the assumption that Q is Reedy cofibrant. Since the model category $\mathtt{sCommAlg}_k$ is left proper, this expresses $\mathrm{sk}_n(X) \otimes^{\boldsymbol{L}}_\Delta A$ is a homotopy pushout of simplicial commutative algebras of quasi-finite type. Thus, it suffices to show that, for a diagram $[\, B \leftarrow A \rightarrow C \,]$ of simplicial commutative algebras, if A, B, C are of quasi-finite type, then so is its homotopy pushout. To show this, we may work with commutative DG algebras instead. One can then apply Lemma 11.1.3 for three times to resolve this diagram by a diagram $[\, B' \hookleftarrow A' \hookrightarrow C' \,]$, where A' is semi-free of finitely many generators in each degree, and the two maps $A' \hookrightarrow B'$ and $A' \hookrightarrow C'$ are semi-free extensions of finitely many generators. Then the pushout of the diagram represents the homotopy pushout of the original diagram, and is semi-free of finitely many generators in each degree, and hence of quasi-finite type, by condition (2) of Proposition 11.1.1. $\square$

Lemma 11.3.12 implies the following proposition which is the main result of this section.

Proposition 11.3.13. *Let X be a finite simplicial set. Suppose that $A : \Delta \to$ $\mathtt{sCommAlg}_k$ is a Reedy smooth cosimplicial simplicial commutative algebras, then the canonical map*

$$p \,:\, X \otimes^{\boldsymbol{L}}_{\Delta} A \to X \otimes_{\Delta} A$$

is an isomorphism in Ho$(\mathtt{sCommAlg}_k)$.

Proof. The π_0 part of both $X \otimes^{\boldsymbol{L}}_{\Delta} A$ and $X \otimes_{\Delta} A$ are given by $X \otimes_{\Delta} \pi_0(A)$. Therefore, by Proposition 11.3.1 and Lemma 11.3.12, it suffices to show that p is étale. To this end, consider the category of simplicies $\Delta \downarrow X$. Being a left Kan extension, the tensor product $X \otimes_{\Delta} A$ can be written as the colimit of the diagram $(\Delta \downarrow X) \to \Delta \xrightarrow{A} \mathtt{sCommAlg}_k$:

$$X \otimes_{\Delta} A \cong \operatorname*{colim}_{([n],c) \in \Delta \downarrow X} A^n \,.$$

Now, choose any Reedy cofibrant resolution $p : Q \overset{\sim}{\twoheadrightarrow} A$, where $Q : \Delta \to$

$\mathtt{sCommAlg}_k$, and consider the induced map

$$X \otimes_\Delta Q \;\cong\; \operatorname*{colim}_{([n],c)\in\Delta\downarrow X} Q^n \;\xrightarrow{p}\; \operatorname*{colim}_{([n],c)\in\Delta\downarrow X} A^n \;\cong\; X \otimes_\Delta A\,.$$

For each $([n],c) \in \Delta \downarrow X$, any $\bar{k}$-valued augmentation $\varepsilon_x : X \otimes_\Delta A_{\bar{k}} \to \bar{k}$ induces an augmentation (still denoted as ε_x) of the simplicial commutative algebra $A_{\bar{k}}^{(n)}$. This determines a map $p^\bullet : Q_{\bar{k}}^\bullet \to A_{\bar{k}}^\bullet$ of $(\Delta \downarrow X)$-diagrams of augmented simplicial commutative algebras. By (the proof of) [77, Proposition 16.3.12], both of $Q_{\bar{k}}^\bullet$ and $A_{\bar{k}}^\bullet$ are still Reedy smooth when considered as $(\Delta \downarrow X)$-diagrams of simplicial commutative algebras. Therefore, by Lemma 11.3.8(2), both of $\mathcal{Q}_{p^*x}(Q_{\bar{k}}^\bullet)$ and $\mathcal{Q}_x(A_{\bar{k}}^\bullet)$ are Reedy cofibrant as $(\Delta \downarrow X)$-diagrams of simplicial k-modules. Moreover, by Lemma 11.3.8(1), the linearization map

$$\mathcal{Q}_{p^*x}(Q_{\bar{k}}^\bullet) \;\to\; \mathcal{Q}_x(A_{\bar{k}}^\bullet)$$

is a degreewise quasi-isomorphism. Therefore, $p^\bullet$ induces a quasi-isomorphism on colimits. In other words, for any augmentation $\varepsilon_x : X \otimes_\Delta A_{\bar{k}} \to \bar{k}$, we have a quasi-isomorphism

$$\mathcal{Q}_x(p) \,:\, \mathcal{Q}_{p^*x}(X \otimes_\Delta Q_{\bar{k}}) \;\to\; \mathcal{Q}_x(X \otimes_\Delta A_{\bar{k}})\,.$$

Since X is finite, $X \otimes_\Delta A$ is a smooth simplicial commutative algebra. Therefore, by Lemma 11.3.8(1), the linearization $\mathcal{Q}_x(X \otimes_\Delta A_{\bar{k}})$ appearing as the target of the map $\mathcal{Q}_x(p)$ represents its derived linearization. By cofibrancy of Q and hence $X \otimes_\Delta Q$, the same is true for the domain of this map. This shows that $p : X \otimes_\Delta^{\boldsymbol{L}} A \to X \otimes_\Delta A$ is étale, which completes the proof. $\square$

The above argument can be modified to show that, in some cases, to compute homotopy colimit, it suffices to resolve a diagram by a Reedy smooth diagram.

To this end, we adopt the terminology of [47], and say that a category I is *very small* if

1. I has a finite set of objects and a finite set of morphisms, and
2. there exists an integer N such that, if $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n$ is a string of composable morphisms of I with $n > N$, then some f_i is the identity morphism in I.

For any object $\alpha \in I$ in such a category, define its degree as the largest integer n such that there exists a string $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n = \alpha$ of composable morphisms, with none of the morphisms f_i being the identity. Then, every non-identity morphism raises degree. Therefore, the category I is a Reedy category, with $I^+ = I$, and I^- consisting only of the identity morphisms.

Proposition 11.3.14. *Let I be a very small category, and let P be an I-shaped diagram of simplicial commutative algebras. Suppose that P is Reedy smooth, then the canonical map* $\mathrm{hocolim}(P) \to \mathrm{colim}(P)$ *is an isomorphism in Ho*$(\mathtt{sCommAlg}_k)$.

Proof. First, we show that both $\mathrm{hocolim}(P)$ and the ordinary colimit $\mathrm{colim}(P)$ are of quasi-finite type. The fact that $\mathrm{colim}(P)$ is of quasi-finite type follows from condition (3) of Proposition 11.1.1. To see that $\mathrm{hocolim}(P)$ is of quasi-finite type, we first construct inductively a cofibrant resolution Q of P such that each $Q(\alpha) \in \mathtt{sCommAlg}_k$ is degreewise finitely generated. Then, $\mathrm{hocolim}(P) \simeq \mathrm{colim}(Q)$ is again of quasi-finite type by condition (3) of Proposition 11.1.1. This allows one to apply Proposition 11.3.1 to the map $f : \mathrm{colim}(Q) \to \mathrm{colim}(P)$. Any augmentation $x : \mathrm{colim}(P)_{\bar{k}} \to \bar{k}$ determine a compatible system of augmentations $x_\alpha : P(\alpha)_{\bar{k}} \to \bar{k}$, which can be further pulled back to a compatible system of augmentations $f^*(x_\alpha) : Q(\alpha)_{\bar{k}} \to \bar{k}$. Taking linearizations gives a map

$$f : (\mathcal{Q}_x(Q(\alpha)_{\bar{k}}))_{\alpha \in I} \to (\mathcal{Q}_x(P(\alpha)_{\bar{k}}))_{\alpha \in I}$$

The assumption that $P \in \texttt{sCommAlg}_k^I$ is Reedy smooth implies (see Lemma 11.3.8) that this map is a termwise quasi-equivalence map of Reedy cofibrant I-shaped diagram in $\texttt{sMod}_{\bar{k}}$. Therefore, f induces a quasi-isomorphism on colimits: i.e., the map $\mathcal{Q}_x(f) : \mathcal{Q}_{f^*x}(\mathrm{colim}(Q)) \to \mathcal{Q}_x(\mathrm{colim}(P))$ is a quasi-isomorphism. By smoothness of $\mathrm{colim}(P)$ and cofibrancy of $\mathrm{colim}(Q)$, this map represents a map on derived linearizations as well. This shows that $\mathrm{colim}(Q) \to \mathrm{colim}(P)$ is étale, finishing the proof. $\square$

CHAPTER 12

DERIVED REPRESENTATION SCHEMES OF SIMPLICIAL GROUPS

12.1 Definition of derived representation schemes

Let G be an affine algebraic group scheme over k. By definition, G is given by the representable functor from the category of commutative k-algebras to the category of groups:

$$G(\,\text{–}\,)\,:\,\mathtt{CommAlg}_k\,\to\,\mathtt{Gr}\,,\quad A\mapsto G(A)\,. \tag{12.1.1}$$

The algebra $\mathcal{O}(G)$ that represents (12.1.1) is called the coordinate ring of G: it is a commutative Hopf algebra with coproduct $\Delta\,:\,\mathcal{O}(G)\,\to\,\mathcal{O}(G)\otimes\mathcal{O}(G)$, $f\,\mapsto\,f_{(1)}\otimes f_{(2)}$, corresponding to the multiplication in G. The functor (12.1.1) has a left adjoint functor which we denote by

$$(\,\text{–}\,)_G\,:\,\mathtt{Gr}\to\mathtt{CommAlg}_k\,,\quad \Gamma\mapsto\Gamma_G\,. \tag{12.1.2}$$

We call (12.1.2) the *representation functor* in G, and think of the commutative algebra Γ_G assigned to a discrete group Γ as the coordinate ring $\mathcal{O}[\mathrm{Rep}_G(\Gamma)]$ of the affine scheme $\mathrm{Rep}_G(\Gamma)$ parametrizing the representations of Γ in G.

It is easy to see that the algebra Γ_G has the following explicit presentation

$$\Gamma_G=\mathrm{Sym}_k[\,k[\Gamma]\otimes_k\mathcal{O}(G)\,]/\langle\!\langle R\rangle\!\rangle\ ,$$

where the relations R are given by

$$\gamma\otimes f_1f_2-(\gamma\otimes f_1)\cdot(\gamma\otimes f_2)\,,\quad\forall\,\gamma\in\Gamma\,,\,\forall\,f_1\,,f_2\in\mathcal{O}(G)\,,$$

$$\gamma_1\gamma_2\otimes f-(\gamma_1\otimes f_{(1)})\cdot(\gamma_2\otimes f_{(2)})\,,\quad\forall\,\gamma_1\,,\gamma_2\in\Gamma\,,\,\forall\,f\in\mathcal{O}(G)\,,$$

$$\gamma\otimes 1-1\,,\quad e_\Gamma\otimes f-f(e_G)\cdot 1\,,\quad\forall\,\gamma\in\Gamma\,,\,\forall\,f\in\mathcal{O}(G)\,.$$

Now, to construct the derived representation functor we embed the category of groups into the category $\mathtt{sGr}$ of simplicial groups and extend the functor (12.1.2) to $\mathtt{sGr}$ in the natural way, assigning to a simplicial group $\Gamma_* : \Delta^{\mathrm{op}} \to \mathtt{Gr}$ the simplicial commutative algebra $(\Gamma_*)_G : \Delta^{\mathrm{op}} \to \mathtt{Gr} \to \mathtt{CommAlg}_k$. We will keep the notation $(-)_G$ for this extended representation functor:

$$(-)_G : \mathtt{sGr} \to \mathtt{sCommAlg}_k . \tag{12.1.3}$$

Lemma 12.1.4. *The functor* (12.1.3) *maps weak equivalences between cofibrant objects in* $\mathtt{sGr}$ *to weak equivalences in* $\mathtt{sCommAlg}_k$*, and hence has a total left derived functor*

$$\boldsymbol{L}(-)_G : \mathit{Ho}(\mathtt{sGr}) \to \mathit{Ho}(\mathtt{sCommAlg}_k) . \tag{12.1.5}$$

Proof. Suppose that $f : \Gamma \to \Gamma'$ is a weak equivalence between cofibrant simplicial groups. Since sGr is a fibrant model category, Γ and Γ' are both fibrant-cofibrant objects. By Whitehead's Theorem, the map f has then a homotopy inverse $g : \Gamma' \to \Gamma$, such that $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$. Now, any homotopy between fibrant-cofibrant objects can be realized using a good cylinder object in $\mathtt{sGr}$. Since sGr is a simplicial model category, there is a natural choice of good cylinder objects for Γ and Γ': namely, $\Gamma \sqcup \Gamma \to \Gamma \times \Delta[1] \to \Gamma$, and similarly for Γ'. For such cylinder objects, the simplicial homotopies (see [112, Def. 5.1]) can be defined by explicit combinatorial relations which are obviously preserved by the functor $(-)_G$. Thus, we conclude that the morphism $g_G : \Gamma'_G \to \Gamma_G$ is a homotopy inverse of $f_G : \Gamma_G \to \Gamma'_G$ in $\mathtt{sCommAlg}_k$ and hence f_G and g_G are mutually inverse isomorphisms in $\mathrm{Ho}(\mathtt{sCommAlg}_k)$. The existence of the derived functor (12.1.5) follows now from [47, Prop. 9.3]. $\square$

Remark 12.1.6. Despite the fact that the representation functor (12.1.3) is a left adjoint functor, it is *not* left Quillen. Indeed, by definition, any left Quillen functor preserves cofibrations; in particular, maps cofibrant objects to cofibrant. To see that this is not the case for (12.1.3) take $G = \mathbb{G}_m$, the multiplicative group, and apply (12.1.3) to the free group on one generator $\Gamma = \mathbb{F}_1$, viewed as a discrete simplicial group in $\mathtt{sGr}$. The result is $\Gamma_G \cong k[x, x^{-1}]$, which is *not* a cofibrant simplicial algebra in $\mathtt{sCommAlg}_k$. Thus, the simplicial adjunction $(-)_G : \mathtt{sGr} \rightleftarrows \mathtt{sCommAlg}_k : G(-)$ is *not* a Quillen pair, so the result of Lemma 12.1.4 is not automatic.

For a fixed simplicial group $\Gamma \in \mathtt{sGr}$, we define $\mathrm{DRep}_G(\Gamma) := \boldsymbol{L}(\Gamma)_G$ and call $\mathrm{DRep}_G(\Gamma)$ the *derived representation scheme* of Γ in the algebraic group G. By definition, $\mathrm{DRep}_G(\Gamma)$ is a simplicial commutative algebra viewed as an object of $\mathrm{Ho}(\mathtt{sCommAlg}_k)$. Following [23, 21], we refer to the corresponding homotopy groups as the *representation homology* of Γ in G :

$$\pi_*[\mathrm{DRep}_G(\Gamma)] := H_*(N[\mathrm{DRep}_G(\Gamma)])$$

It is easy to check that the functor (12.1.3) commutes with π_0, so that for any $\Gamma \in \mathtt{sGr}$, there is a natural isomorphism in $\mathtt{CommAlg}_k$:

$$\pi_0[\mathrm{DRep}_G(\Gamma)] \cong [\pi_0(\Gamma)]_G \tag{12.1.7}$$

Hence, for an ordinary group $\Gamma \in \mathtt{Gr}$ (viewed as a discrete simplicial group), we have

$$\pi_0[\mathrm{DRep}_G(\Gamma)] \cong \Gamma_G = \mathcal{O}[\mathrm{Rep}_G(\Gamma)] \ .$$

This justifies our notation and terminology for $\mathrm{DRep}_G(\Gamma)$. We record one useful property of the derived representation functor that will play a role in computations in Section 7.

Lemma 12.1.8. *The functor* $\mathrm{DRep}_G : Ho(\mathtt{sGr}) \to Ho(\mathtt{sCommAlg}_k)$ *preserves homotopy pushouts.*

Proof. Suppose we are given a diagram $[\Gamma' \hookleftarrow \Gamma \hookrightarrow \Gamma'']$ of simplicial groups, where Γ is semi-free, and the two arrows are semi-free extensions. Let P be the diagram $P = [(\Gamma')_G \leftarrow (\Gamma)_G \rightarrow (\Gamma'')_G]$ obtained from $[\Gamma' \hookleftarrow \Gamma \hookrightarrow \Gamma'']$ by applying the representation functor (12.1.3). Then we have to show that the canonical map $\mathrm{hocolim}(P) \to \mathrm{colim}(P)$ is an isomorphism in $\mathrm{Ho}(\mathtt{sCommAlg}_k)$. Since the weak equivalences in $\mathtt{sCommAlg}_k$ are preserved under filtered colimits, it suffices to assume that the simplicial groups Γ, Γ' and Γ'' are of finite type (*i.e.*, they are semi-free with finitely many nondegenerate generators). While P is not, in general, Reedy cofibrant, it is Reedy smooth for any algebraic group G. The result therefore follows from Proposition 11.3.14. $\square$

12.2 Realization functor

In this section, we give another construction of representation homology that does not use the Kan equivalence (4.1.2). Our starting point is a general categorical principle which asserts that any left adjoint functor on the category of simplicial sets with values in a (cocomplete) category $\mathscr{C}$ arises from a cosimplicial object in $\mathscr{C}$. Specifically, a cosimplicial object $F : \Delta \to \mathscr{C}$ gives rise to the simplicial adjunction

$$- \otimes_\Delta F : \mathtt{sSet} \rightleftarrows \mathscr{C} : \mathrm{Hom}_{\mathscr{C}}(F, -)\,,$$

where $(- \otimes_\Delta F)$ denotes the left Kan extension of F along the Yoneda embedding $Y : \Delta \to \mathtt{sSet}$, and $\mathrm{Hom}_{\mathscr{C}}(F, -)$ is the functor assigning to $A \in \mathrm{Ob}(\mathscr{C})$ the simplicial set $\{\mathrm{Hom}_{\mathscr{C}}(F([n]), A)\}_{n \geq 0}$. This gives an equivalence between the category $\mathscr{C}^\Delta$ of cosimplicial objects in $\mathscr{C}$ and the category of simplicial adjunctions with values in $\mathscr{C}$ (see, e.g., [79, Prop. 3.1.5]).

The fundamental example is the cosimplicial space $\Delta^* \in \mathtt{Top}^{\Delta}$ defined by the geometric simplices $\{\Delta^n\}_{n\geq 0}$. Under the above equivalence, it corresponds to the classical adjunction between simplicial sets and topological spaces:

$$|-|:\ \mathtt{sSet}\ \rightleftarrows\ \mathtt{Top}\ :\mathrm{Hom}_{\mathtt{Top}}(\Delta^*, -)\ ,$$

where $|-| = - \otimes_{\Delta} \Delta^*$ is the geometric realization functor defined in Section 2.2. In general, it is therefore natural to think of functors of the form $(- \otimes_{\Delta} F)$: $\mathtt{sSet} \to \mathscr{C}$ as *realization functors* of simplicial sets in $\mathscr{C}$.

Now, let us take an affine algebraic group G and consider its classifying space BG. This is naturally a simplicial affine k-scheme, and its coordinate ring $\mathcal{O}(BG) : \Delta \to \mathtt{CommAlg}_k$ is a cosimplicial commutative k-algebra. The next proposition shows that the 'realization' functor corresponding to $\mathcal{O}(BG)$ is just the classical representation functor $(-)_G$ (see Section 4.2).

Proposition 12.2.1. *For any $X \in \mathtt{sSet}_0$, there is a natural isomorphism of commutative algebras*

$$X \otimes_{\Delta} \mathcal{O}(BG) \cong \pi_1(X)_G = \mathcal{O}[\mathrm{Rep}_G(\pi_1(X))]$$

In particular, if Γ is a discrete group, then $\mathrm{B}\Gamma \otimes_{\Delta} \mathcal{O}(BG) \cong \Gamma_G$ *.*

Proof. For any $A \in \mathtt{CommAlg}_k$, we have canonical isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathtt{CommAlg}_k}(X \otimes_{\Delta} \mathcal{O}(BG), A) &\cong \mathrm{Hom}_{\mathtt{sSet}}(X, \mathrm{Hom}_{\mathtt{CommAlg}_k}(\mathcal{O}(BG), A)) \\
&\cong \mathrm{Hom}_{\mathtt{sSet}_0}(X, BG(A)) \\
&\cong \mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}X, G(A)) \\
&\cong \mathrm{Hom}_{\mathtt{Gr}}(\pi_1(X), G(A)) \\
&\cong \mathrm{Hom}_{\mathtt{CommAlg}_k}(\pi_1(X)_G, A)\ .
\end{aligned} \tag{12.2.2}$$

Here, the third and fourth isomorphisms follow from the fact that $G(A)$ is a *discrete* simplicial group, so that $BG(A) = \overline{W}G(A)$ and $\mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}X, G(A)) = \mathrm{Hom}_{\mathtt{Gr}}(\pi_0(\mathbb{G}X), G(A))$. The desired proposition follows now from Yoneda Lemma. $\square$

Proposition 12.2.1 suggests that one could define representation homology of spaces in terms of the (non-abelian) derived tensor product $\otimes^{\boldsymbol{L}}_{\Delta}$. This is indeed possible, and is the content of the following

Theorem 12.2.3. *For any* $X \in \mathtt{sSet}_0$*, there is a natural isomorphism in* $Ho(\mathtt{sCommAlg}_k)$:

$$\mathrm{DRep}_G(\mathbb{G}X) \,\cong\, X \overset{\boldsymbol{L}}{\underset{\Delta}{\otimes}} \mathcal{O}(BG)\,. \tag{12.2.4}$$

We wish to prove this theorem by constructing a series of adjunction isomorphisms similar to those in (12.2.2). Notice that in (12.2.2), the series of adjunctions end with the set $\mathrm{Hom}_{\mathtt{CommAlg}_k}(\pi_1(X)_G, A)$, which in turn determines the commutative algebra $\pi_1(X)_G$ by Yoneda lemma if we allow A to vary over all commutative algebras. Now, if we wish to determine the simplicial commutative algebra $\mathbb{G}(X)_G$, it suffices to consider $\mathrm{Hom}_{\mathtt{CommAlg}_k}(\mathbb{G}(X)_G, A)$, which is now a cosimplicial set since we take the Hom set in each simplicial degree. The idea then is to run the adjunctions (12.2.2) backwards, starting from this cosimplicial set. Indeed, by the definition of the representation functor $(-)_G$, one immediately have an isomorphism of cosimplicial sets

$$\mathrm{Hom}_{\mathtt{CommAlg}_k}(\mathbb{G}(X)_G, A) \,\cong\, \mathrm{Hom}_{\mathtt{Gr}}(\mathbb{G}(X), G(A)) \tag{12.2.5}$$

which is the first step of such backward adjunction isomorphisms.

However, it is already not straightforward to proceed to the next step. In (12.2.2), we made use of the adjunction $(\mathbb{G}, \overline{W})$ between simplicial groups and reduced simplicial sets to obtain the isomorphism $\mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}X, G(A)) \cong \mathrm{Hom}_{\mathtt{sSet}_0}(X, BG(A))$. In the present case, we have instead the cosimplicial set $\mathrm{Hom}_{\mathtt{Gr}}(\mathbb{G}X, G(A))$ obtained by taking the Hom set degreewise. The adjunction $(\mathbb{G}, \overline{W})$ is therefore not directly applicable.

To proceed, we write the cosimplicial set $\mathrm{Hom}_{\mathtt{Gr}}(\mathbb{G}X, G(A))$ in a different way. Suppose that we have a cosimplicial simplicial group $RG(A)$, then in each cosimplicial degree of $RG(A)$, one can take the Hom set $\mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}X, RG(A))$ over the category $\mathtt{sGr}$ of simplicial groups, this gives a cosimplicial set $\mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}X, RG(A))$. Our next goal is to construct a cosimplicial simplicial group $RG(A)$ out of the group $G(A)$, such that we have an isomorphism of cosimplicial sets:

$$\mathrm{Hom}_{\mathtt{Gr}}(\mathbb{G}X, G(A)) \cong \mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}X, RG(A)) \tag{12.2.6}$$

The construction of $RG(A)$ is rather formal, so it is helpful to consider the adjunction (12.2.6) in a more general context. Thus, we consider simplicial groups $\Gamma \in \mathtt{sGr}$, which will play the role of $\mathbb{G}(X)$, and discrete group $H \in \mathtt{Gr}$, which will play the role of $G(A)$, and try to construct a cosimplicial simplicial group $RH \in (\mathtt{sGr})^{\Delta}$, such that one has an isomorphism of cosimplicial sets

$$\mathrm{Hom}_{\mathtt{Gr}}(\Gamma, H) \cong \mathrm{Hom}_{\mathtt{sGr}}(\Gamma, RH) \tag{12.2.7}$$

natural in all $\Gamma \in \mathtt{sGr}$ and $H \in \mathtt{Gr}$.

To avoid confusion, we will write the cosimplicial degree as a superscript, and the simplicial degree as a subscript. Thus, for example, for each fixed cosimplicial degree n, one has a simplicial group $RH^n = RH^n_*$. It is then clear that the isomorphism (12.2.7) actually determines RH. Indeed, in cosimplicial degree n, we must have

$$\mathrm{Hom}_{\mathtt{sGr}}(\Gamma, RH^n) \;=\; \mathrm{Hom}_{\mathtt{sGr}}(\Gamma, RH)^n \;\cong\; \mathrm{Hom}_{\mathtt{Gr}}(\Gamma, H)^n \;=\; \mathrm{Hom}_{\mathtt{Gr}}(\Gamma_n, H)$$

Thus, we see that the functor $H \mapsto RH^n$ which sends the group $H \in \mathtt{Gr}$ to our hypothetical $RH^n \in \mathtt{sGr}$ should be right adjoint to the functor $\Gamma \mapsto \Gamma_n$ which sends a simplicial group $\Gamma \in \mathtt{sGr} = \mathtt{Gr}^{\Delta^{\mathrm{op}}}$. to the group in degree n.

Consider the inclusion map $* \hookrightarrow \Delta^{\mathrm{op}}$ that sends the one-point category $*$ to the object $[n] \in \Delta^{\mathrm{op}}$. Then the functor $\Gamma \mapsto \Gamma_n$ we considered above is the restriction functor $\mathtt{Gr}^{\Delta^{\mathrm{op}}} \to \mathtt{Gr}$ with respect to the inclusion $* \hookrightarrow \Delta^{\mathrm{op}}$ of indexing categories. Therefore, this restriction functor $\Gamma \mapsto \Gamma_n$ has a right adjoint given by right Kan extension. By standard construction, if $RH^n_* \in \mathtt{Gr}^{\Delta^{\mathrm{op}}}$ is this right Kan extension applied to $H \in \mathtt{Gr}$, then we have

$$RH^n_m \;\cong\; \lim_{[m]\downarrow *} H \;\cong\; \prod_{\mathrm{Hom}_{\Delta^{\mathrm{op}}}([m],[n])} H \;=\; H^{\times \mathrm{Hom}_{\Delta}([n],[m])} \tag{12.2.8}$$

where we have denoted by $H^{\times S}$ the product of copies of H indexed by the set S, which is covariant in H and contravariant in S. Hence, the product $H^{\times \mathrm{Hom}_{\Delta}([n],[m])}$ is covariant in $[n] \in \Delta$ and contravariant in $[m] \in \Delta$, which shows that the formula (12.2.8) does define a cosimplicial simplicial group RH.

Thus, we take (12.2.8) as the definition of RH. By construction, we have for each n a natural isomorphism $\mathrm{Hom}_{\mathtt{sGr}}(\Gamma, RH^n) \cong \mathrm{Hom}_{\mathtt{Gr}}(\Gamma_n, H)$ of sets, which resembles to the isomorphism (12.2.7) of cosimplicial sets. Now, apply this construction RH to the group $H = G(A)$, we have $R(G(A)) = G(A)^{\times \mathrm{Hom}_{\Delta}([n],[m])}$. This group has an alternative description as the group of A-valued points of the algebraic group $G^{\times \mathrm{Hom}_{\Delta}([n],[m])}$. Thus, if we define the cosimplicial simplicial *algebraic* group $RG^n_m := G^{\times \mathrm{Hom}_{\Delta}([n],[m])}$, then we have $R(G(A)) = (RG)(A)$, which will be henceforth denoted as $RG(A)$.

Thus, our construction has given us a cosimplicial simplicial algebraic group RG whose (cosimplicial simplicial) group $RG(A)$ of A-valued points satisfies the adjunction isomorphism (12.2.6). Now consider the functor $\overline{W} : \mathtt{sGr} \to \mathtt{sSet}_0$, which can be applied to the cosimplicial simplicial group $RG(A)$, the result is the cosimplicial simplicial set $\overline{W}(RG(A)) \in (\mathtt{sSet}_0)^\Delta$. Recall that, for each simplicial group Γ, the (reduced) simplicial set $\overline{W}(\Gamma)$ is defined (see, e.g., [67, Section V.4]) by

$$\overline{W}(\Gamma)_n = \Gamma_n \times \Gamma_{n-1} \times \ldots \times \Gamma_0$$

Since only products are involved, this formula is well-defined when Γ is a simplicial algebraic group. Thus, one can consider the cosimplicial simplicial affine scheme $\overline{W}RG$, whose (cosimplicial simplicial) set of A-valued points again coincide with the previously defined version : $(\overline{W}RG)(A) = \overline{W}(RG(A))$, which will be denoted as $\overline{W}RG(A)$.

We may take the ring of functions on the cosimplicial simplicial affine scheme $\overline{W}RG$. The result is a cosimplicial simplicial commutative algebra $\mathcal{O}(\overline{W}RG) \in (\mathtt{sCommAlg}_k)^\Delta$. With this, we are ready to state the following

Proposition 12.2.9. *For each simplicial set* $X \in \mathtt{sSet}$, *we have an isomorphism of simplicial commutative algebras*

$$X \underset{\Delta}{\otimes} \mathcal{O}(\overline{W}RG) \cong \mathbb{G}(X/X_0)_G$$

where $X/X_0 \in \mathtt{sSet}_0$ *is the reduced simplicial set obtained by contracting the set* X_0 *of vertices of* X *to a single point.*

Proof. For any $A \in \mathtt{CommAlg}_k$, we have the following series of canonical isomorphisms of cosimplicial sets:

$$\begin{aligned}
\mathrm{Hom}_{\mathtt{CommAlg}_k}(X \otimes_\Delta \mathcal{O}(\overline{W}RG), A) &\cong \mathrm{Hom}_{\mathtt{sSet}}(X, \mathrm{Hom}_{\mathtt{CommAlg}_k}(\mathcal{O}(\overline{W}RG), A)) \\
&\cong \mathrm{Hom}_{\mathtt{sSet}}(X, \overline{W}RG(A)) \\
&\cong \mathrm{Hom}_{\mathtt{sSet}_0}(X/X_0, \overline{W}RG(A)) \\
&\cong \mathrm{Hom}_{\mathtt{sGr}}(\mathbb{G}(X/X_0), RG(A)) \\
&\cong \mathrm{Hom}_{\mathtt{Gr}}(\mathbb{G}(X/X_0), G(A)) \\
&\cong \mathrm{Hom}_{\mathtt{Gr}}(\mathbb{G}(X/X_0)_G, G(A)) \ .
\end{aligned} \tag{12.2.10}$$

Here the fifth isomorphism follows from (12.2.6), while the sixth isomorphism follows from (12.2.5). □

In view of Proposition 11.3.13, in order to prove Theorem 12.2.3, it suffices to show the following two lemmas:

Lemma 12.2.11. *The cosimplicial simplicial commutative algebra $\mathcal{O}(\overline{W}RG) \in (\mathtt{sCommAlg}_k)^\Delta$ is weakly equivalent to $\mathcal{O}(BG) \in (\mathtt{CommAlg}_k)^\Delta \subset (\mathtt{sCommAlg}_k)^\Delta$. More precisely, there is a map $\mathcal{O}(\overline{W}RG) \to \mathcal{O}(BG)$ of cosimplicial simplicial commutative algebras that gives a weak equivalence of simplicial commutative algebras in each cosimplicial degree.*

Proof. Consider the cosimplicial simplicial affine scheme $\overline{W}RG$. By definition, the cosimplicial affine scheme $(\overline{W}RG)_m = (\overline{W}RG)^*_m$ in simplicial degree m is given by

$$(\overline{W}RG)_m = RG_m \times RG_{m-1} \times \ldots \times RG_0$$

By the definition (12.2.8) applied to $H = G$, we therefore have

$$(\overline{W}RG)^n_m = G^{\times \mathrm{Hom}_\Delta([n],[m])} \times G^{\times \mathrm{Hom}_\Delta([n],[m-1])} \times \ldots \times G^{\times \mathrm{Hom}_\Delta([n],[0])}$$

If we take the ring of functions on both sides, the simplicial and cosimplicial degrees of the left hand side will be switched, and we have

$$\mathcal{O}(\overline{W}RG)^m_n = \mathcal{O}(G)^{\otimes \mathrm{Hom}_\Delta([n],[m])} \otimes \mathcal{O}(G)^{\otimes \mathrm{Hom}_\Delta([n],[m-1])} \otimes \ldots \otimes \mathcal{O}(G)^{\otimes \mathrm{Hom}_\Delta([n],[0])}$$

Consider each of the tensor component $\mathcal{O}(G)^{\otimes \mathrm{Hom}_\Delta([n],[k])}$ of the right hand side of this equation. If we fix k and let the simplicial degree n vary, we see that this is precisely the simplicial commutative algebra

$$\mathcal{O}(G)^{\otimes \mathrm{Hom}_\Delta(-,[k])} \;=\; \mathcal{O}(G) \otimes \Delta^k$$

obtained by tensoring the commutative algebra $\mathcal{O}(G)$ with the simplicial set $\Delta^k = \mathrm{Hom}_\Delta(-,[k])$. Combining this with the above equation for $\mathcal{O}(\overline{W}RG)^m_n$, we have

$$\begin{aligned}\mathcal{O}(\overline{W}RG)^m_* \;&=\; \mathcal{O}(G)^{\otimes \mathrm{Hom}_\Delta(-,[m])} \otimes \mathcal{O}(G)^{\otimes \mathrm{Hom}_\Delta(-,[m-1])} \otimes \ldots \otimes \mathcal{O}(G)^{\otimes \mathrm{Hom}_\Delta(-,[0])} \\ &=\; (\, \mathcal{O}(G) \otimes \Delta^m \,) \,\otimes\, (\, \mathcal{O}(G) \otimes \Delta^{m-1} \,) \,\otimes\, \ldots \,\otimes\, (\, \mathcal{O}(G) \otimes \Delta^0 \,)\end{aligned} \tag{12.2.12}$$

for each fixed cosimplicial degree m.

Now, the cosimplicial commutative algebra $\mathcal{O}(BG) \in (\mathtt{CommAlg}_k)^\Delta \subset (\mathtt{sCommAlg}_k)^\Delta$ is given by

$$\mathcal{O}(BG)^m \;=\; \mathcal{O}(G) \otimes \overset{m}{\cdots} \otimes \mathcal{O}(G) \;=(\, \mathcal{O}(G) \otimes \Delta^0 \,) \otimes (\, \mathcal{O}(G) \otimes \Delta^0 \,) \otimes \ldots \otimes (\, \mathcal{O}(G) \otimes \Delta^0 \,) \tag{12.2.13}$$

Comparing (12.2.12) and (12.2.13), one sees immediately that there is a weak equivalence $\mathcal{O}(\overline{W}RG)^m \to \mathcal{O}(BG)^m$ of simplicial commutative algebra defined by sending each tensor component $\mathcal{O}(G) \otimes \Delta^k$ in (12.2.12) to a tensor component $\mathcal{O}(G) \otimes \Delta^0$ in (12.2.13). A straightforward verification shows that this map of diagrams is functorial in the cosimplicial degree $[m] \in \Delta$, and hence defines the desired map $\mathcal{O}(\overline{W}RG) \to \mathcal{O}(BG)$ of cosimplicial simplicial commutative algebras. $\square$

Lemma 12.2.14. *The diagram $\mathcal{O}(\overline{W}RG) \in (\mathtt{sCommAlg}_k)^{\Delta}$ is Reedy smooth.*

Proof. To prove this, it suffices to show that if $X \hookrightarrow Y$ is a cofibration of simplicial set, then the induced map

$$X \underset{\Delta}{\otimes} \mathcal{O}(\overline{W}RG) \to Y \underset{\Delta}{\otimes} \mathcal{O}(\overline{W}RG) \tag{12.2.15}$$

of simplicial commutative algebras is a smooth extension. Since weak equivalences are preserved under filtered colimits in the category of simplicial commutative algebras, it suffices to assume that X and Y are finite simplicial sets.

By Proposition 12.2.9, the map (12.2.15) is isomorphic to the map $\mathbb{G}(X/X_0)_G \to \mathbb{G}(Y/Y_0)_G$. Since $X/X_0 \to Y/Y_0$ is still a cofibration of reduced simplicial sets, the map $\mathbb{G}(X/X_0) \to \mathbb{G}(Y/Y_0)$ is a semi-free extension of simplicial groups. This shows that the induced map $\mathbb{G}(X/X_0)_G \to \mathbb{G}(Y/Y_0)_G$ is a smooth extension of simplicial commutative algebras, which completes the proof. $\square$

Part III

Perverse sheaves and knot contact homology

CHAPTER 13

INTRODUCTION

In a series of papers [120, 121, 122, 123, 124], L. Ng introduced and studied a new algebraic invariant of a link L in $\mathbb{R}^3$ represented by a semi-free differential graded (DG) algebra $\mathcal{A}_L$. The structure of this DG algebra (termed *op. cit.* a combinatorial knot DGA) is determined by an element of a braid group B_n representing the link L. The homology of $\mathcal{A}_L$ is called the *knot contact homology* $HC_*(L)$ as it coincides with the Legendrian contact homology[1] of the unit conormal bundle $\Lambda_L \subseteq ST^*\mathbb{R}^3$ of L. This coincidence was conjectured in [120, 121] and proved later in [54, 55], where it was shown, in fact, that the entire combinatorial knot DGA is isomorphic to a geometrically defined DG algebra computing the Legendrian contact homology of Λ_L.

Our original motivation was to understand Ng's combinatorial proof of the invariance of $\mathcal{A}_L$ (up to stable isomorphism) under the Markov moves. We should remark that, although the differential of $\mathcal{A}_L$ is defined in [120] by an explicit formula, its combinatorial structure is fairly complicated and its algebraic origin seems mysterious. Even the fact that the 0-th homology of $\mathcal{A}_L$ is a link invariant is far from being obvious from the definition of [120] (*cf.* [120, Section 4.3]). As a result, we have come up with a different, more conceptual construction that makes the Markov invariance of $\mathcal{A}_L$ quite transparent[2] and, more importantly, places knot contact homology in one row with other classical invariants, such as knot groups and Alexander modules.

To clarify the ideas we begin by recalling a classical theorem of E. Artin and

[1] in the sense of [57] (see also [52, 53])

[2] In fact, the invariance of our construction under type I Markov moves follows directly from its definition.

J. Birman that gives a natural presentation of the link group $\pi_1(\mathbb{R}^3\setminus L)$ in terms of a braid representing L. Let $\mathscr{D}$ be the unit disk in $\mathbb{R}^2$, and let $\{p_1,\dots,p_n\}\subset\mathscr{D}$ be a set of distinct points in the interior of D. It is well known that the braid group on n-strands, B_n, can be identified with the mapping class group of $(\mathscr{D}\setminus\{p_1,\dots,p_n\},\partial\mathscr{D})$, and as such it acts naturally on the fundamental group $\pi_1(\mathscr{D}\setminus\{p_1,\dots,p_n\},p_0)$, where $p_0\in\mathscr{D}\setminus\{p_1,\dots,p_n\}$ is a basepoint (which we choose near the boundary of $\mathscr{D}$). The fundamental group $\pi_1(\mathscr{D}\setminus\{p_1,\dots,p_n\},p_0)$ is a free group $\mathbb{F}_n$ of rank n based on generators $x_1,\dots,x_n$ which correspond to small loops in $\mathscr{D}\setminus\{p_1,\dots,p_n\}$ around the points p_i. Explicitly, in terms of these generators, the action of B_n on $\pi_1(\mathscr{D}\setminus\{p_1,\dots,p_n\},p_0)\cong\mathbb{F}_n$ is given by

$$\sigma_i\ :\ \begin{cases} x_i & \mapsto\ \ x_i\,x_{i+1}\,x_i^{-1} \\ x_{i+1} & \mapsto\ \ x_i \\ x_j & \mapsto\ \ x_j \quad (j\neq i,i+1) \end{cases} \tag{13.0.1}$$

where σ_i $(i=1,2,\dots,n-1)$ are the standard generators of B_n. This action is usually called the *Artin representation* as it provides a faithful realization of B_n as a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$. Now, the Artin-Birman Theorem (see [12, Theorem 2.2]) asserts that the fundamental group of the complement of the link $L=\hat{\beta}\subset\mathbb{R}^3$ corresponding to a braid $\beta\in B_n$ has the presentation

$$\pi_1(\mathbb{R}^3\setminus L)\ \cong\ \langle x_1,\,x_2,\,\dots\,,\,x_n\,|\,\beta(x_1)=x_1,\,\beta(x_2)=x_2,\,\dots,\,\beta(x_n)=x_n\rangle\ , \tag{13.0.2}$$

where $\beta(x_i)$ denotes the action of β on x_i via the Artin representation.

We abstract this situation in the following way. Let $\mathcal{C}$ be a category with finite colimits. We assume that we are given a family of braid group actions $\varrho_n: B_n\to\mathrm{Aut}\,A^{(n)}$, $n\geq 1$, on objects of $\mathcal{C}$ having the properties:

(1) For each $n\geq 1$, $A^{(n)}$ is the n-fold coproduct of one and the same object A of $\mathcal{C}$.

(2) The actions ϱ_n are *local* and *homogeneous* in the sense that each $\sigma_i \in B_n$ acts only on the $(i, i+1)$-copy of $A^{(2)}$ in $A^{(n)}$ while keeping the rest fixed, and any two standard generators of B_n act in the same way on the corresponding copies of $A^{(2)}$ for all $n \geq 1$.

Such braid group actions are determined (generated) by a single morphism $\sigma : A \amalg A \to A \amalg A$ in the category $\mathcal{C}$ that we call a *cocartesian Yang-Baxter operator* (*cf.* Definition 14.0.1). For example, the Artin representations are generated by a cocartesian Yang-Baxter operator in the category of groups given by $\sigma : \mathbb{F}_1 \amalg \mathbb{F}_1 \to \mathbb{F}_1 \amalg \mathbb{F}_1,\ x_1 \mapsto x_1 x_2 x_1^{-1}, x_2 \mapsto x_1$.

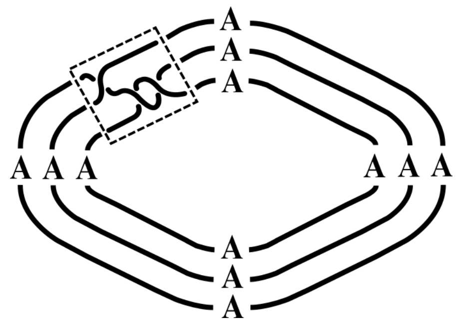
Now, for an arbitrary cocartesian Yang-Baxter operator (A, σ), we define a universal construction $\mathcal{L}(A, \sigma)$ that associates to each braid $\beta \in B_n$ the coequalizer of the endomorphisms id and β of the object $A^{(n)}$, or equivalently, the following pushout in $\mathcal{C}$:

$$\mathcal{L}(A,\sigma)[\beta] := \mathrm{coeq}\big[A^{(n)} \underset{\mathrm{id}}{\overset{\beta}{\rightrightarrows}} A^{(n)}\big] \;=\; \mathrm{colim}\,[\,A^{(n)} \xleftarrow{(\beta,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)}\,]\,. \tag{13.0.3}$$

We call (13.0.3) the *categorical closure of the braid β on the object A* with respect to the Yang-Baxter operator σ. This terminology can be justified by the following "picture"[3] of the pushout (13.0.3) that manifestly exhibits it as "a braid closure

[3]In fact, this "picture" of a categorical braid closure can be formalized by using the diagrammatic tensor calculus developed by A. Joyal, R. Street and others (see [87, 88, 141]). We briefly discuss it at the end of Section 14.

on A'':



$$\operatorname{colim}\left[\, A^{(3)} \xleftarrow{(\beta,\,\mathrm{id})} A^{(3)} \amalg A^{(3)} \xrightarrow{(\mathrm{id},\,\mathrm{id})} A^{(3)} \,\right]$$

In the case of Artin representations, the Artin-Birman Theorem (13.0.2) implies that $\mathcal{L}(\mathbb{F}_1, \sigma)[\beta] \cong \pi_1(\mathbb{R}^3 \backslash L)$. This means, in particular, that $\mathcal{L}(\mathbb{F}_1, \sigma)[\beta]$ is a link invariant.

In general, we show that, if a cocartesian Yang-Baxter operator (A, σ) satisfies some natural conditions, which we call the *Reidemeister conditions* (see Definition 14.0.15), then the isomorphism class of the categorical closure of any braid with respect to (A, σ) is stable under the Markov moves, and hence defines a link invariant (*cf.* Theorem 14.0.10 and Theorem 14.0.17). Apart from the group $\pi_1(\mathbb{R}^3 \setminus L)$, many classical link invariants arise in this way (see, for example, Theorem 14.0.8 that represents as a categorical braid closure the Alexander module).

Next, we consider the category $\mathsf{Perv}(D, \{p_1, \dots, p_n\})$ of perverse sheaves on the disk D with only possible singularities at the points $\{p_1, \dots, p_n\}$. In [66], Gelfand, MacPherson and Vilonen showed that $\mathsf{Perv}(D, \{p_1, \dots, p_n\})$ is equivalent to the category $\tilde{\mathscr{Q}}^{(n)}$ of finite-dimensional k-linear representations of the following quiver

$$Q^{(n)} \;=\; \begin{tikzcd} n \arrow[dr, bend left, "a_n"] & 1 \arrow[d, bend left, "a_1"] & 2 \arrow[dl, bend left, "a_2"] \\ & 0 \arrow[ul, bend left, "a_n^*"] \arrow[u, bend left, "a_1^*"] \arrow[ur, bend left, "a_2^*"] \arrow[dr, bend left, "a_3^*"] & \\ \ddots & & 3 \arrow[ul, bend left, "a_3"] \end{tikzcd}$$

such that the operators $T_i := e_0 + a_i a_i^*$ act as isomorphisms for all $i = 1, 2, \dots, n$. More formally, $\tilde{\mathscr{Q}}^{(n)}$ can be described as the category $\mathtt{Mod}\,\tilde{A}^{(n)}$ of finite-dimensional modules over the k-category

$$\tilde{A}^{(n)} := k\langle Q^{(n)}\rangle[T_1^{-1}, \dots, T_n^{-1}] \tag{13.0.4}$$

which is obtained by localizing the path category of $Q^{(n)}$ at the set of morphisms $\{T_1, \dots, T_n\}$. Now, the braid group B_n acts on the disk D with n marked points $\{p_1, \dots, p_n\}$ as a mapping class group, and this naturally induces an action on the category $\mathsf{Perv}(D, \{p_1, \dots, p_n\})$. It was shown in [66] that, under the equivalence $\mathsf{Perv}(D, \{p_1, \dots, p_n\}) \simeq \tilde{\mathscr{Q}}^{(n)}$, the action of B_n on the category of perverse sheaves corresponds to a *strict* action on the category $\tilde{\mathscr{Q}}_n$ (*cf.* [66, Proposition 1.3]). This, in turn, induces an action of B_n on the k-category $\tilde{A}^{(n)}$, which is given explicitly (on generating morphisms of $\tilde{A}^{(n)}$) by the following formulas

$$\sigma_i \,:\, \begin{cases} a_i \;\mapsto\; T_i\, a_{i+1} & \\ a_{i+1} \;\mapsto\; a_i & \\ a_j \;\mapsto\; a_j & (j \neq i, i+1) \\ a_i^* \;\mapsto\; a_{i+1}^*\, T_i^{-1} & \\ a_{i+1}^* \;\mapsto\; a_i^* & \\ a_j^* \;\mapsto\; a_j^* & (j \neq i, i+1) \end{cases} \tag{13.0.5}$$

We call (13.0.5) the *Gelfand-MacPherson-Vilonen (GMV) braid action*.

The GMV braid actions are generated by a single cocartesian Yang-Baxter operator in the category of (small) *pointed* k-categories $\mathtt{Cat}_k^*$. Specifically, for each $n \geq 1$, the k-category $\tilde{A}^{(n)}$ is the coproduct (fusion product) in $\mathtt{Cat}_k^*$ of n copies of the k-category $\tilde{A} = k\langle Q\rangle[T^{-1}]$, where $k\langle Q\rangle$ is the path category of the quiver $Q = \big[\,1 \underset{a^*}{\overset{a}{\rightleftarrows}} 0\,\big]$ with the distinguished object 0. The corresponding Yang-Baxter operator $\sigma : \tilde{A} \amalg \tilde{A} \to \tilde{A} \amalg \tilde{A}$ is given by

$$(a_1,\, a_1^*) \;\mapsto\; (T_1\, a_2,\, a_2^*\, T_1^{-1})\;, \quad (a_2,\, a_2^*) \;\mapsto\; (a_1,\, a_1^*)\;. \tag{13.0.6}$$

Just as in the case of Artin actions, it is easy to check that (13.0.6) satisfies the Reidemeister conditions, and hence the categorical braid closure with respect to $(\tilde{A}, \sigma)$ is a link invariant. For a given $\beta \in B_n$, this invariant is represented by the equivalence class of the k-category $\tilde{A}_L := \mathcal{L}(\tilde{A}, \sigma)[\beta]$, which we call the *(fully noncommutative) link k-category*[4] of $L = \hat{\beta}$. In Section 19, we will show that the k-category $\tilde{A}_L$ is a natural extension of the fully noncommutative cord algebra of [54, 124] in the sense that the latter can be identified with the endomorphism algebra of an object in $\tilde{A}_L$. Thus, in our algebraic formalism, the link category $\tilde{A}_L$ arises exactly the same way as the link group $\pi_1(\mathbb{R}^3 \setminus L)$, provided we take as an input the Gelfand-MacPherson-Vilonen braid action instead of the Artin representation.

At this point, we pause to remark that the notion of a categorical braid closure has already appeared in the literature: explicitly – in the case of groups (see [165, 44]), and in a somewhat disguised form, in the theory of quandles (see, for example, [65, 45]). From this last perspective, our results give a precise interpretation of such geometric knot invariants as a cord algebra in combinatorial terms of racks and quandles (see Remark 14.0.18 below).

[4]Strictly speaking, the categorical braid closure gives a specialization of the fully noncommutative link k-category, with all longitude parameters set to be 1 (see Remark 19.0.9). For a general definition of $\tilde{A}_L$, we refer to Section 19, Definition 19.0.1.

However, our main observation is that the simple categorical formalism we outlined above admits an interesting generalization to homotopical contexts. Specifically, if the category $\mathcal{C}$ that we work with has a natural class $\mathcal{W}$ of weak equivalences (e.g., $\mathcal{C}$ is a Quillen model category or a homotopical category in the sense of [51]), then the operation of a categorical braid closure is usually not invariant under weak equivalences, *i.e.* it is not well-defined[5] in the homotopy category $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$. In abstract homotopy theory, there is a standard way to remedy this problem: namely, replace a homotopy non-invariant functor F by its derived functor, which gives a universal approximation to F on the level of homotopy categories (see, e.g., [47, Section 9]). In our situation, we can define a "derived" version of the categorical braid closure by simply replacing the 'colim' in the definition (13.0.3) by its derived functor: the homotopy colimit '$\mathrm{hocolim}$'. To be precise, given a cocartesian Yang-Baxter operator (A, σ) in (say) a model category $\mathcal{C}$, we define the *homotopy braid closure* of $\beta \in B_n$ with respect to (A, σ) by

$$h\mathcal{L}(A, \sigma)[\beta] := \mathrm{hocolim}\,[\, A^{(n)} \xleftarrow{(\beta,\,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\,\mathrm{id})} A^{(n)} \,]\,. \tag{13.0.7}$$

One of our main results (Theorem 15.0.4) states that if (A, σ) satisfies the Reidemeister conditions (and A is flat in an appropriate sense), then the weak equivalence class of the homotopy braid closure on A, *i.e.* the isomorphism class of (13.0.7) in the homotopy category $\mathrm{Ho}(\mathcal{C})$, is invariant under the Markov moves, and hence defines a link invariant. This last invariant is more refined than the one given by the usual categorical braid closure in the same way as the homotopy type of a topological space is a more refined invariant of the space than just its fundamental group.

[5]The problem is that pointwise weak equivalences of diagrams do not necessarily induce weak equivalences of colimits, so the objects defined by colimits of diagrams defined up to homotopy are not well-defined, even up to homotopy type.

Now, let us return to our basic example of the cocartesian Yang-Baxter operator $(\tilde{A}, \sigma)$ associated to the GMV action, see (13.0.6). To define the homotopy braid closure with respect to this operator we will regard the k-category $\tilde{A}$ as an object of the category $\mathtt{dgCat}_k^*$ comprising all (small) pointed DG categories. The category $\mathtt{dgCat}_k^*$ has a natural model structure, in which the weak equivalences are the quasi-equivalences[6] of DG categories (see [152]). Its homotopy theory has been extensively studied in recent years with a view towards applications in algebraic geometry and representation theory (see, e.g., [92, 155] and references therein).

The homotopy braid closure of the GMV action in the model category $\mathtt{dgCat}_k^*$ gives a new link invariant, which is a quasi-equivalence class of DG categories. For a given $\beta \in B_n$, formula (13.0.7) allows us, in fact, to construct an explicit representative for the corresponding quasi-equivalence class that we call the *fully noncommutative link DG category* $\tilde{\mathcal{A}}_L$ (see Definition 19.0.1). If we assume, for simplicity, that L is a knot (*i.e.*, a link with a single component), then $\tilde{\mathcal{A}}_L$ contains a distinguished object, and the endomorphism DG algebra of that object is isomorphic to the fully noncommutative knot DGA constructed in [54]. This observation is part of Theorem 19.0.4 that we state in full generality (for links with an arbitrary number of components) but do not prove in this thesis. Instead, we sketch a proof of an analogous result – Theorem 17.0.6 – that identifies the *framed* knot DGA (originally introduced in [122]) with the DG endomorphism algebra of a distinguished object in the homotopy braid closure of a modified GMV action. The modification amounts to collapsing all objects of the GMV k-category $\tilde{A}^{(n)}$, except for the base object '0', to a single object '1',

[6] Recall that a *quasi-equivalence* of DG categories is a DG functor $F : \mathscr{A} \to \mathscr{B}$ such that $F : \mathscr{A}(X,Y) \to \mathscr{B}(FX,FY)$ is a quasi-isomorphism of k-complexes for all objects $X, Y \in \mathrm{Ob}(\mathscr{A})$ and the induced functor on the 0th homology $H_0(F) : H_0(\mathscr{A}) \xrightarrow{\sim} H_0(\mathscr{B})$ is an equivalence of categories.

while preserving all the generating morphisms a_i and a_i^*. We also impose n extra relations $a_i^* \, a_i = (\mu - 1) \, e_1$, one for each $i = 1, 2, \ldots, n$, that depend on an invertible central parameter μ in the ground ring k. The resulting k-category $A^{(n)}$ with two objects $\{0, 1\}$ inherits the GMV braid action (13.0.5), and one can still define its homotopy braid closure by formula (13.0.7)[7].

Now, as in the case of topological spaces, to compute the homotopy colimit of a diagram like (13.0.7) one should first 'resolve' the objects by their cofibrant models $R \stackrel{\sim}{\twoheadrightarrow} A^{(n)}$, then replace one of the arrows by a (weakly equivalent) cylinder cofibration, and then take the usual colimit in the underlying category $\mathcal{C}$:

$$\operatorname{hocolim} [\, A^{(n)} \leftarrow A^{(n)} \amalg A^{(n)} \to A^{(n)}] \;\cong\; \operatorname{colim} [\, R \leftarrow R \amalg R \hookrightarrow \operatorname{Cyl}(R) \,] \,. \quad (13.0.8)$$

In the category of DG categories with finitely many objects, there is a canonical cylinder object $\operatorname{Cyl}_{\mathrm{BL}}(R)$ defined for any semi-free DG category R. We call this object the *Baues-Lemaire cylinder* as it was originally constructed (in the case of chain DG algebras) in [13]. The differentials of $\operatorname{Cyl}_{\mathrm{BL}}(R)$ are defined by explicit formulas in terms of differentials of R, while for a semi-free resolution $R \stackrel{\sim}{\twoheadrightarrow} A^{(n)}$, the differentials of R are determined explicitly by the relations of $A^{(n)}$. Thus, taking the colimit (13.0.8) with the help of the Baues-Lemaire cylinder $\operatorname{Cyl}_{\mathrm{BL}}(R)$, we find an explicit presentation for the knot DG category $\mathcal{A}$, given in Definition 17.0.3. An elementary calculation then shows that the DG algebra $\mathcal{A}(1,1)$ consisting of all endomorphisms of the object '1' in the DG k-category $\mathcal{A}$ is precisely the knot DGA defined in [122]. This explains the 'mysterious' algebraic formula for the differentials in Ng's combinatorial knot DGA: it arises

[7] To introduce the second central parameter $\lambda \in k^{\times}$ we also modify the arrow (β, id) in the homotopy colimit (13.0.7) by appropriately twisting the action map $\beta : A^{(n)} \to A^{(n)}$ (see Section 16).

from the Baues-Lemaire cylinder on the natural DG resolution of the k-category $A^{(n)}$.

In [121, 122], Ng has also given an explicit description of the 0th homology of his knot DGA in terms of the knot group $\pi_1(\mathbb{R}^3 \setminus K)$ and the peripheral pair (m, l) of a meridian and longitude in $\pi_1(\mathbb{R}^3 \backslash K)$. We extend this description to the 0th homology of the knot DG category, both in the framed and fully noncommutative cases (see Theorem 18.0.2 and Theorem 19.0.7). Our proof is purely algebraic, in contrast to a topological proof given in [121, 122].

Finally, we mention one interesting application of our results that brings us back to topology. Given a link $L \subset \mathbb{R}^3$, we consider the category $\mathsf{Perv}(\mathbb{R}^3, L)$ of perverse sheaves on $\mathbb{R}^3$ constructible with respect to the stratification $L \hookrightarrow \mathbb{R}^3 \hookleftarrow \mathbb{R}^3 \setminus L$ with perversity given by $p(1) = 0$ and $p(3) = -1$. Our Theorem 19.0.10 states that $\mathsf{Perv}(\mathbb{R}^3, L)$ is equivalent to the category of finite-dimensional left modules over the fully noncommutative link k-category $\tilde{A}_L$. This leads to an algebraic description of the category $\mathsf{Perv}(\mathbb{R}^3, L)$ in terms of groups and quivers, similar, in spirit, to the Gelfand-MacPherson-Vilonen description of the category $\mathsf{Perv}(D, \{p_1, \ldots, p_n\})$.

Part III of this thesis is organized as follows. In Section 14, we define cocartesian Yang-Baxter operators and the associated categorical braid closure, and give two criteria – the Wada condition (Definition 14.0.9) and the Reidemeister condition (Definition 14.0.15) – for the categorical braid closure to be a link invariant (Theorem 14.0.10 and Theorem 14.0.17). In Section 15, we extend the construction of a categorical braid closure to the homotopical setting. The main result in this section is Theorem 15.0.4. In Section 16, we introduce our main example of the cocartesian Yang-Baxter operator associated to the GMV braid action. In Section 17, we calculate the homotopy braid closure with respect to the GMV operator, and show that the resulting DG category is an extension of Ng's knot DGA (see Theorem 17.0.5 and Theorem 17.0.6). The main tool in this calculation is the Baues-Lemaire cylinder on a semi-free DG category; for reader's convenience, we review its construction is some detail. In Section 18, we compute the 0th homology of the knot DG category, called the *knot k-category*, and give a description of this category in terms of the knot group together with a peripheral pair (see Theorem 18.0.2). In Section 19, we define the fully noncommutative link DG category and extend the main results of Sections 17 and 18 to this case (see Theorem 19.0.4 and Theorem 19.0.7). While the input for the knot DG category introduced in Sections 17 and 18 is the modified GMV action, the input for the fully noncommutative case is the original GMV action. This allows us to relate the corresponding module category to perverse sheaves (see Theorem 19.0.10). Finally, in Section 20, we give two natural generalizations of the GMV operator, inspired by the work of Wada [165] and Crisp-Paris [44] in the group case. These generalizations satisfy the Reidemeister conditions, and hence the corresponding homotopy braid closures give link invariants generalizing the link DG category associated to the original GMV action. We will discuss these new link invariants elsewhere.

CHAPTER 14

YANG-BAXTER OPERATORS AND CATEGORICAL BRAID CLOSURE

Let $\mathcal{C}$ be a category closed under finite colimits. Let $A \in \mathcal{C}$ be an object of $\mathcal{C}$. For an integer $n \geq 2$, we denote the n-fold coproduct of copies of A in $\mathcal{C}$ by $A^{(n)} := A \amalg \overset{n}{\cdots} \amalg A$. If $f : A \to B$ is a morphism in $\mathcal{C}$, we denote its n-fold coproduct by $f^{(n)} : A^{(n)} \to B^{(n)}$. Now, suppose that we are given an object A and a morphism $\sigma : A \amalg A \to A \amalg A$ in $\mathcal{C}$. Then, for each $n \geq 2$ and $i = 1, 2 \ldots, n-1$, σ induces a morphism $\sigma_{i,i+1} : A^{(n)} \to A^{(n)}$ defined by

$$\sigma_{i,i+1} := \mathrm{id}^{(i-1)} \amalg \sigma \amalg \mathrm{id}^{(n-i-1)} : A^{(n)} \to A^{(n)}$$

Definition 14.0.1. A *cocartesian Yang-Baxter operator* on A is an invertible morphism

$$\sigma : A \amalg A \to A \amalg A$$

satisfying the equation

$$\sigma_{23}\, \sigma_{12}\, \sigma_{23} = \sigma_{12}\, \sigma_{23}\, \sigma_{12} \qquad \text{in} \quad \mathrm{Hom}_{\mathcal{C}}(A^{(3)}, A^{(3)})\,. \tag{14.0.2}$$

We will often use the term “Yang-Baxter” as an adjective for an invertible morphism σ satisfying (14.0.2).

Any cocartesian Yang-Baxter operator σ on A extends in a natural way to a *left* action of the Artin braid group B_n on $A^{(n)}$ for each $n \geq 2$. We refer to this action as the action generated by σ.

We give two basic examples of cocartesian Yang-Baxter operators corresponding to two classical representations of the braid group B_n.

Example 14.0.3. Let $\mathcal{C} = \texttt{Gr}$ be the category of groups, and let $A = \mathbb{F}_1 \in \mathcal{C}$ be the free group on one generator. Consider the map $\sigma : A \amalg A \to A \amalg A$ given by

$$\sigma : \mathbb{F}_2 \to \mathbb{F}_2 \qquad x_1 \mapsto x_1 x_2 x_1^{-1}, \qquad x_2 \mapsto x_1$$

As mentioned in the Introduction, this is a Yang-Baxter map generating the Artin representations[1].

Example 14.0.4. Let $\mathcal{C} = \texttt{Mod}(R)$ be the category of modules over the commutative ring $R = \mathbb{Z}[t, t^{-1}]$. Take $A = R$ to be the free R-module of rank one, and define the map $\sigma : R^{\oplus 2} \to R^{\oplus 2}$ by left multiplication by the matrix $\begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}$. This map is a cocartesian Yang-Baxter operator in the category $\texttt{Mod}(R)$ generating the classical (unreduced) Burau representations.

Now, given a cocartesian Yang-Baxter operator $\sigma : A \amalg A \to A \amalg A$, we denote the resulting braid group action on the n-fold coproduct by

$$\phi_n^{(A,\sigma)} : B_n \to \operatorname{Aut} A^{(n)}$$

Abusing notation, for a braid $\beta \in B_n$, we will often write the automorphism $\phi_n^{(A,\sigma)}(\beta)$ simply as β if the underlying Yang-Baxter operator is understood to be (A, σ).

Definition 14.0.5. The *categorical braid closure* of a braid $\beta \in B_n$ with respect to a cocartesian Yang-Baxter operator $\sigma : A \amalg A \to A \amalg A$ is defined to be the coequalizer

$$\mathcal{L}(A,\sigma)[\beta] := \operatorname{coeq}\big[\, A^{(n)} \underset{\mathrm{id}}{\overset{\beta}{\rightrightarrows}} A^{(n)} \,\big]\ ,$$

[1]In the literature (see, e.g., [12]), it is more common to extend σ to a *right* braid action. Thus, if $\Phi : B_n \to B_n$ is the anti-isomorphism of B_n where $\Phi(\sigma_i) = \sigma_i$, then the automorphism in the convention of [12] corresponding to the element $\beta \in B_n$ is equal to the automorphism in our present convention corresponding to the element $\Phi(\beta) \in B_n$.

or equivalently, the following pushout in $\mathcal{C}$:

$$\mathcal{L}(A,\sigma)[\beta] = \operatorname{colim}\left[\, A^{(n)} \xleftarrow{(\beta,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)} \,\right].$$

Recall that the coequalizer of two morphisms $f : X \to Y$ and $g : X \to Y$ in $\mathcal{C}$ is an object $E \in \mathcal{C}$ given together with a moprhism $p : Y \to E$ such that the pair (E, p) is universal among all pairs satisfying $pf = pg$. In practice, computing the coequalizer amounts to taking a quotient of the object Y by the relations $f(x) = g(x)$ for all $x \in X$. Thus, in Example 14.0.3, the categorical closure of $\beta \in B_n$ is the group presented by

$$\mathcal{L}(A,\sigma)[\beta] = \langle\, x_1, \ldots, x_n \mid \beta(x_1) = x_1, \ldots, \beta(x_n) = x_n \,\rangle\, .$$

The next theorem is a classical result first stated by E. Artin in [1] and proved by J. Birman in [12].

Theorem 14.0.6 (Artin-Birman)**.** *The categorical closure of a braid $\beta \in B_n$ with respect to the Artin representation is the fundamental group of the link complement $\mathbb{R}^3 \backslash L$, where $L = \hat{\beta}$ is the closure of the braid β.*

Similarly, in Example 14.0.4, the categorical closure of $\beta \in B_n$ is the module over $R = \mathbb{Z}[t, t^{-1}]$ given by

$$\mathcal{L}(R,\sigma)[\beta] = \operatorname{coker}[R^{\oplus n} \xrightarrow{\mathrm{id}-\beta} R^{\oplus n}]\, . \tag{14.0.7}$$

In this case, we have the following theorem due to D. Goldschmidt [76].

Theorem 14.0.8. *The categorical closure of a braid with respect to the Burau action is the Alexander module of the unlinked disjoint union $L \cup O$ of the braid closure $L = \hat{\beta}$ with the unknot O.*

Thus, the categorical braid closure of both the Artin and Burau examples are link invariants. This raises the natural question: When does the categorical braid closure of a cocartesian Yang-Baxter operator produce a link invariant? To address this question we begin with the following definition.

Given a map $\sigma : A \amalg A \to A \amalg A$, we consider the coequalizer

$$E := \mathrm{coeq}\big[\, A \overset{\sigma\circ i_2}{\underset{i_2}{\rightrightarrows}} A \amalg A \,\big]$$

where $i_2 : A \to A \amalg A$ is the canonical map identifying A with its second copy in $A \amalg A$. We let $p : A \amalg A \to E$ denote the universal map such that $p \circ \sigma \circ i_2 = p \circ i_2$.

Definition 14.0.9. We say that the map σ is *Wada* if the following composition is an isomorphism in $\mathcal{C}$:

$$j' : A \xrightarrow{i_1} A \amalg A \xrightarrow{p} E$$

For a Wada map σ, we consider the map

$$j : A \xrightarrow{i_1} A \amalg A \xrightarrow{\sigma} A \amalg A \xrightarrow{p} E$$

and define the *torsion* of σ to be the map

$$\chi(\sigma) = (j')^{-1} \circ j : A \to A$$

We say that a Wada map σ has *trivial torsion* if $\chi(\sigma) = \mathrm{id}_A$ is the identity map.

As an easy exercise for the reader, we recommend to check that both the Artin and Burau Yang-Baxter operators have trivial torsion. The next theorem explains why the categorical braid closures of these operators give link invariants.

Theorem 14.0.10 (Wada)**.** *Suppose that a cocartesian Yang-Baxter operator $\sigma : A \amalg A \to A \amalg A$ is Wada with trivial torsion, then the isomorphism type of the categorical braid closure is invariant under Markov moves, and hence give a link invariant.*

This theorem was proved in [165] in the special case when $\mathcal{C}$ is the category $\mathtt{Gr}$ of groups, and the object $A \in \mathtt{Gr}$ is the free group $\mathbb{F}_1$ on one generator. However, the arguments of [165] can be easily formalized and extended to a proof in the general case.

The Wada condition involves coequalizers, which makes its verification somewhat clumsy in practice (especially, in the homotopical setting which we will discuss in the next section). We therefore introduce another condition on σ that, among other things, turns the Wada condition into a simpler form.

Definition 14.0.11. We say that a map $\sigma : A \amalg A \to A \amalg A$ is *dualizable* if

1. The map $\sigma : A \amalg A \to A \amalg A$ is invertible.
2. The map $\sigma^R_U = (\sigma \circ i_2, i_2) : A \amalg A \to A \amalg A$ is invertible.
3. The map $\sigma^L_U = (\sigma \circ i_1, i_1) : A \amalg A \to A \amalg A$ is invertible.

With this definition, we have

Proposition 14.0.12. *Assume that $\sigma : A \amalg A \to A \amalg A$ is dualizable. Consider the composition of maps*

$$(j, j') : A \amalg A \xrightarrow{\sigma^L_U} A \amalg A \xrightarrow{(\sigma^R_U)^{-1}} A \amalg A \xrightarrow{\nabla} A\,, \tag{14.0.13}$$

where ∇ is the canonical folding map. Then, σ is Wada if and only if the map j' is an isomorphism. In this case, the torsion of σ is given by

$$\chi(\sigma) = (j')^{-1} \circ j \tag{14.0.14}$$

Next, we introduce our main definition.

Definition 14.0.15. We say a map $\sigma : A \amalg A \to A \amalg A$ is *Reidemeister* if it is Yang-Baxter, dualizable, and Wada with invertible torsion $\chi(\sigma)$ (see Proposition 14.0.12).

In Theorem 14.0.10, we required the torsion of σ to be trivial. However, in our main example of the cocartesian Yang-Baxter operator associated to the GMV action (see Section 16,) the torsion is not trivial, but invertible. It turns out, if σ is Reidemiester, with not necessarily trivial torsion, then an analogue of Theorem 14.0.10 holds, provided we modify the categorical braid closure in an appropriate way. From now on, for simplicity, we will work with knots (*i.e.*, links with one component), while all that follows holds for the general case of links (See [11] for more details).

Let $\sigma : A \amalg A \to A \amalg A$ be a Reidemeister operator with (invertible) torsion $\chi : A \to A$. Suppose that $\beta \in B_n$ is a braid that closes to a knot $\hat{\beta} = K$. Write $w = w(\beta)$ for the writhe of β.

Definition 14.0.16. The *normalized categorical closure* of β with respect to σ is defined by

$$\bar{\mathcal{L}}(A, \sigma)[\beta] = \mathrm{coeq}\big[\, A^{(n)} \underset{\mathrm{id}}{\overset{\Psi_0}{\rightrightarrows}} A^{(n)} \,\big]\ ,$$

where Ψ_0 is the composition of morphisms: $A^{(n)} \xrightarrow{\chi^{-w} \amalg \mathrm{id}^{(n-1)}} A^{(n)} \xrightarrow{\beta} A^{(n)}$.

Theorem 14.0.17. *For any Reidemeister operator $\sigma : A \amalg A \to A \amalg A$, the isomorphism type of the normalized categorical braid closure is invariant under Markov moves, and hence gives a knot invariant.*

There is a conceptual way to prove Theorems 14.0.10 and 14.0.17 by interpreting the categorical braid closure as an abstract trace in the sense of [88].

To this end, we extend the category $\mathcal{C}$ to a larger category $\widehat{\mathcal{C}}$ with the same object set $\mathrm{Ob}(\widehat{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$. The hom-set $\mathrm{Hom}_{\widehat{\mathcal{C}}}(A, B)$ in $\widehat{\mathcal{C}}$ is given by the set of cospans $[B \to X \leftarrow A]$ modulo isomorphisms that are identity on A and B. The composition in $\widehat{\mathcal{C}}$ is defined by pushouts in the obvious way. The category $\widehat{\mathcal{C}}$ has a monoidal product $\boxtimes$ induced by the coproduct in $\mathcal{C}$; hence the monoidal structure of $\widehat{\mathcal{C}}$ extends the cocartesian monoidal structure on $\mathcal{C}$ in the sense that there is a faithful, strongly monoidal functor $\iota : (\mathcal{C}, \amalg) \hookrightarrow (\widehat{\mathcal{C}}, \boxtimes)$. Moreover, the monoidal category $(\widehat{\mathcal{C}}, \boxtimes)$ has a canonical pivotal structure, and therefore an abstract trace axiomatized in [88] and [151]. This construction allows us to interpret the categorical braid closure as an abstract trace in $\widehat{\mathcal{C}}$, and the Wada condition (Definition 14.0.9) can then be interpreted as a condition on the partial trace of σ viewed as a morphism in $\widehat{\mathcal{C}}$ under the faithful embedding $\iota : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$. We remark that the Wada condition reinterpreted this way is analogous to a condition on partial trace for "enriched Yang-Baxter operators" introduced by Turaev in [161]. This interpretation allows us to prove Theorem 14.0.10 and Theorem 14.0.17 by diagrammatic tensor calculus. In fact, starting with a Reidemeister operator, one can construct a ribbon category (in the sense of [141]), whose associated link invariant, which lives in the set $\mathrm{Hom}_{\widehat{\mathcal{C}}}(\phi, \phi)$ of isomorphism classes of objects in $\mathcal{C}$, coincides with the categorical braid closure. One can see this as another justification for the term "categorical braid closure". For details, we refer the reader to [11].

Remark 14.0.18. The notion of a cocartesian Yang-Baxter operator is closely related to biracks and biquandles[2]. To make this relation precise we recall the (dual) Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathtt{Set})$ for a category $\mathcal{C}$ that associates to an object $A \in \mathcal{C}$ the corepresentable functor $h^A := \mathrm{Hom}(A, -) : \mathcal{C} \to \mathtt{Set}$. A cobirack structure on A can then be defined by factoring h^A into a composition

[2]For the definition and basic examples of biracks and biquandles we refer to [65, 45].

of functors $\mathcal{C} \to \mathsf{Birack} \xrightarrow{\text{forget}} \mathtt{Set}$. Similarly, a cobiquandle structure on A is the factorization of h^A into a composition of functors $\mathcal{C} \to \mathsf{Biquandle} \xrightarrow{\text{forget}} \mathtt{Set}$. Then, one can show that, giving a cobirack structure on A is equivalent to giving a dualizable cocarteisan Yang-Baxter operator on A. Similarly, giving a cobiquandle structure on A is equivalent to giving a Reidemeister operator on A with trivial torsion. Thus, in particular, given a Reidemeister operator on A with trivial torsion, the set $X_B = \mathrm{Hom}_{\mathcal{C}}(A, B)$ has a natural biquandle structure for any object $B \in \mathcal{C}$. In fact, many examples of biquandles in the literature arise in this manner. (In particular, almost all examples of biquandles given in [65] are of this form.)

Given a biquandle X and a link L, one can define a combinatorial link invariant called $\mathrm{Col}_X(L)$, which is the set of colorings of a link diagram of L by the biquandle X (see [65]). If the link L is the closure of a braid $\beta \in B_n$, and if the biquandle $X = X_B$ arises from a cobiquandle structure on an object A in the sense above, then we have

$$\mathrm{Col}_{X_B}(L) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}(A, \sigma)[\beta], B)\ .$$

This last formula gives a combinatorial interpretation of the categorical braid closure in terms of arcs of a link diagram that we alluded to in the Introduction.

CHAPTER 15

HOMOTOPY BRAID CLOSURE

In this section, we will work with model categories and assume the reader to have some familiarity with the theory of model categories and derived functors. For an excellent introduction, we recommend the Dwyer-Spalinski article [47], which covers enough material for Part III. For a more comprehensive study of model categories, we refer to [77, 79].

If $\mathcal{C}$ is a model category, then the notion of a categorical braid closure associated to a cocartesian Yang-Baxter operator in $\mathcal{C}$ admits a natural generalization, which is obtained by replacing colimits in Definition 14.0.5 and Definition 14.0.16 by homotopy colimits. The analogues of Theorem 14.0.10 and Theorem 14.0.17 hold in this homotopical setting, provided that the object A satisfies a pseudoflatness condition (which roughly says that the n-fold homotopy coproduct of A coincides with the n-fold coproduct of A). Since the translation to the homotopical context is fairly straightforward, we omit here formal details. Instead, we will give explicit definitions and statements only in the case, where the notion of a normalized braid closure is further refined by allowing the knot in question to be colored by certain maps.

Definition 15.0.1. Given a cocartesian Yang-Baxter operator $\sigma : A \amalg A \to A \amalg A$, we say that a map $\theta : A \to A$ is *σ-natural* if the following two diagrams commute

$$\begin{tikzcd} A \amalg A \arrow[r, "\sigma"] \arrow[d, "\theta \amalg \mathrm{id}"'] & A \amalg A \arrow[d, "\mathrm{id} \amalg \theta"] \\ A \amalg A \arrow[r, "\sigma"] & A \amalg A \end{tikzcd} \qquad \begin{tikzcd} A \amalg A \arrow[r, "\sigma"] \arrow[d, "\mathrm{id} \amalg \theta"'] & A \amalg A \arrow[d, "\theta \amalg \mathrm{id}"] \\ A \sqcup A \arrow[r, "\sigma"] & A \amalg A \end{tikzcd}$$

Now, let $\sigma : A \amalg A \to A \amalg A$ be a Reidemeister operator with (invertible) torsion $\chi : A \to A$, and let $\theta : A \to A$ be a σ-natural map. Suppose that $\beta \in B_n$ is a braid that closes to a knot $\hat{\beta} = K$.

Definition 15.0.2. The *θ-colored normalized homotopy closure* of β with respect to σ is defined to be the homotopy coequalizer in $\mathcal{C}$:

$$h\bar{\mathcal{L}}_\theta(A, \sigma)[\beta] = \mathrm{hocoeq}\big[\, A^{(n)} \underset{\mathrm{id}}{\overset{\Psi}{\rightrightarrows}} A^{(n)} \,\big]$$

where $\Psi : A^{(n)} \to A^{(n)}$ is the composition

$$A^{(n)} \xrightarrow{\theta\chi^{-w} \amalg \mathrm{id}^{(n-1)}} A^{(n)} \xrightarrow{\beta} A^{(n)} \tag{15.0.3}$$

When $\theta = \mathrm{id}_A$ is the identity map, which is always σ-natural, this construction obviously reduces to the original one without coloring.

Theorem 15.0.4. *Let $\sigma : A \amalg A \to A \amalg A$ be a Reidemeister operator on a pseudoflat object A in a model category $\mathcal{C}$, and let $\theta : A \to A$ is a σ-natural map. Then, the isomorphism type in $\mathrm{Ho}(\mathcal{C})$ (i.e., the weak equivalence type in $\mathcal{C}$) of the θ-colored normalized homotopy braid closure is invariant under Markov moves, and hence gives a knot invariant.*

This theorem allows one to refine many classical link invariants defined by categorical braid closure. To illustrate this we will return to Examples 14.0.3 and 14.0.4 in Section 14.

Example 15.0.5. To compute the homotopy braid closure of the Burau representations (see Example 14.0.4), we embed the module category $\mathtt{Mod}(R)$ into the category $\mathrm{Ch}(R)$ of chain complexes in the usual way. The module $A = R$ is then identified with the chain complex $A = [0 \to R \to 0]$, with R concentrated in degree 0. The category $\mathrm{Ch}(R)$ has a natural (projective) model structure, with weak equivalences being the quasi-isomorphisms and the fibrations being the degreewise surjective morphisms of complexes (see [79, Section 2.3]). Every object in this model category is pseudoflat. The corresponding homotopy category $\mathrm{Ho}(\mathcal{C})$ is the (unbounded) derived category $\mathcal{D}(R)$ of R-modules. Now, the homotopy closure of a braid $\beta \in B_n$ with respect to the Burau operator $\sigma : R^{\oplus 2} \to R^{\oplus 2}$ is given by the mapping cone of the morphism $\mathrm{id} - \beta$ in the derived category $\mathcal{D}(R)$: *i.e.*,

$$h\mathcal{L}(R,\sigma)[\beta] = \mathrm{C\hat{o}ne}(\mathrm{id} - \beta) \;:=\; [\,0 \;\to\; R^{\oplus n} \;\xrightarrow{\mathrm{id}-\beta}\; R^{\oplus n} \;\to\; 0\,]\,,$$

where the two copies of $R^{\oplus n}$ are concentrated in homological degrees 0 and 1. The isomorphism class of $h\mathcal{L}(R,\sigma)[\beta]$ in $\mathcal{D}(R)$ is a link invariant by Theorem 15.0.4. Note that the homology of this complex in degree 0 is the cokernel of $\mathrm{id} - \beta$, which is precisely the categorical braid closure $\mathcal{L}(R,\sigma)[\beta]$, see (14.0.7).

Example 15.0.6. To compute the homotopy braid closure of the Artin representations (see Example 14.0.3), we embed the category of groups into the model category `sGr` of *simplicial* groups. The model structure on `sGr` is inherited from the category `sSet` of simplicial sets, so that the weak equivalences and fibrations of simplicial groups are the weak equivalences and fibrations of the underlying simplicial sets (see, e.g., [67]). The category of simplicial groups has a rich homotopy theory which is classically known to be equivalent to that of topological spaces. Precisely, there is a Quillen equivalence between `sGr` and the category $\mathtt{Top}_{0,*}$ of connected pointed topological spaces given by the composition of functors

$$\mathtt{sGr} \xrightarrow{\overline{W}} \mathtt{sSet} \xrightarrow{|-|} \mathtt{Top}_{0,*} \tag{15.0.7}$$

where $\overline{W}$ is Kan's bar construction assigning to a simplicial group its classifying simplicial space and $|-|$ is Milnor's geometric realization functor. The functor (15.0.7) induces an equivalence of the homotopy categories $\mathrm{Ho}(\mathtt{sGr}) \cong \mathrm{Ho}(\mathtt{Top}_{0,*})$, which gives a bijective correspondence between the homotopy classes of simplicial groups and the homotopy classes of pointed connected CW complexes (see, e.g., [67, V.6.4]). In this way, every simplicial group can be thought of as representing a topological space (up to homotopy).

Now, if we regard $\mathbb{F}_1$ as a discrete simplicial group in $\mathtt{sGr}$, then the homotopy closure $h\mathcal{L}(\mathbb{F}_1, \sigma)[\beta]$ of a braid $\beta \in B_n$ with respect to the Artin operator $\sigma : \mathbb{F}_2 \to \mathbb{F}_2$ is represented by a simplicial group in $\mathrm{Ho}(\mathtt{sGr})$ that corresponds under the above equivalence to the link complement $\mathbb{R}^3 \backslash L$ of the closure $L = \hat{\beta}$ (*cf.* Part I):

$$|\, \overline{W}\, h\mathcal{L}(\mathbb{F}_1, \sigma)[\beta]\, | \simeq \mathbb{R}^3 \backslash L \,.$$

Thus, in the case of Artin representations, the homotopy braid closure completely recovers the homotopy type of the space $\mathbb{R}^3 \setminus L$, while the categorical braid closure gives only its fundamental group.

Remark 15.0.8. Theorem 15.0.4 holds in a more general case, when various conditions on the map σ hold only up to homotopy. For example, one can require the Yang-Baxter equation (14.0.2) to hold only in $\mathrm{Ho}(\mathcal{C})$, the maps in Definition 14.0.11 to be only weak equivalences and the map $j : A \to A$ defined in (14.0.13) to be only an isomorphism in $\mathrm{Ho}(\mathcal{C})$. The braid group action, as well as the torsion map, are then only defined in $\mathrm{Ho}(\mathcal{C})$. This makes a precise definition of a homotopy braid closure a little tedious. We omit it here, referring the reader to [11] instead.

CHAPTER 16

THE GELFAND-MACPHERSON-VILONEN ACTION

In this section, we fix a commutative ring k with unit. By a *k-category*, we mean a category enriched over the category of k-modules. Let $\mathtt{Cat}_k^*$ be the category of all (small) pointed k-categories, where a k-category $\mathscr{A}$ is *pointed* if there is a distinguished object $* \in \mathscr{A}$. Maps (*i.e.*, k-linear functors) between pointed k-categories are required to preserve the distinguished objects.

As in the Introduction, we consider the path category $k\langle Q\rangle$ of the quiver $Q = \left[1 \underset{a^*}{\overset{a}{\rightleftarrows}} 0 \right]$ Let $T \in k\langle Q\rangle(0,0)$ be the element in the endomorphism algebra of $k\langle Q\rangle$ of the object 0 defined by $T = e_0 + aa^*$, and let $\tilde{A}$ be the k-category $\tilde{A} = k\langle Q\rangle[T^{-1}]$, which is pointed by taking the object 0 as the distinguished object.

The coproduct in $\mathtt{Cat}_k^*$ is given by the *fusion product*. *i.e.* the coproduct of $X, Y \in \mathtt{Cat}_k^*$ is the k-category obtained by collapsing the two distinguished objects in the disjoint union of X and Y into a single object. In particular, the n-fold coproduct of $\tilde{A} \in \mathtt{Cat}_k^*$ is the k-category $\tilde{A}^{(n)}$ defined in (13.0.4).

We begin with the following result mentioned in the Introduction.

Theorem 16.0.1 (Gelfand, MacPherson, Vilonen)**.** *The map $\sigma : \tilde{A}^{(2)} \to \tilde{A}^{(2)}$ defined by* (13.0.6) *is a cocartesian Yang-Baxter operator on the object $\tilde{A}$ in the category $\mathtt{Cat}_k^*$. We call σ the GMV operator.*

The GMV operator induces an action of B_n on $\tilde{A}^{(n)}$, where the generator $\sigma_i \in B_n$ acts on objects by swapping i and $i+1$, while fixing all other objects, and on morphisms by formula (13.0.5). The next observation is straightforward to check.

Lemma 16.0.2. *The GMV operator* (13.0.6) *is Reidemeister with torsion given by*

$$\chi : \tilde{A} \to \tilde{A}, \qquad a \mapsto Ta, \qquad a^* \mapsto a^* T^{-1} \tag{16.0.3}$$

As explained in the Introduction, formula (13.0.5) for the braid group action first appeared in [66] in relation to perverse sheaves. More precisely, it was shown in [66] that any choice of 'cuts' (*i.e.*, a family Θ of n simple curves on $D \setminus \{p_1, \dots, p_n\}$, going from a chosen point near p_i to the chosen endpoint p_0 near the boundary ∂D, so that any two such curves intersect only at p_0) induces an equivalence of categories $\tilde{E}_\Theta : \mathsf{Perv}(D, \{p_1, \dots, p_n\}) \simeq \tilde{\mathscr{Q}}$ from the category of perverse sheaves on the disk D with only possible singularities at the points $\{p_1, \dots, p_n\}$, to a quiver category $\tilde{\mathscr{Q}}$ *isomorphic* to the category $\mathtt{Mod}(\tilde{A}^{(n)})$ of finite-dimensional modules over the k-category $\tilde{A}^{(n)}$.

Now, the braid group B_n acts as a mapping class group on the disk D with n marked points $\{p_1, \dots, p_n\}$, and hence acts (in a certain sense) on the category $\mathsf{Perv}(D, \{p_1, \dots, p_n\})$. If we fix a family Θ of cuts, this translates to an action of B_n on the quiver category $\tilde{\mathscr{Q}}$. In fact, it is shown in [66] that there is a *strict* action of B_n on the quiver category $\tilde{\mathscr{Q}}$ that coincides under the equivalence $\tilde{E}_\Theta$ with the natural action on the category $\mathsf{Perv}(D, \{p_1, \dots, p_n\})$ (see [66, Proposition 1.3]). This strict B_n action on the quiver category $\tilde{\mathscr{Q}}$ is in fact induced by the action (13.0.5) on the k-category $\tilde{A}^{(n)}$. More precisely, the left action (13.0.5) induces a strict left action on the module category $\mathtt{Mod}(\tilde{A}^{(n)})$ where $\beta \in B_n$ acts by $M \mapsto (\beta^{-1})^*(M)$. This coincides under the isomorphism of categories $\mathtt{Mod}(\tilde{A}^{(n)}) \cong \tilde{\mathscr{Q}}$ with the strict B_n action on the quiver category $\tilde{\mathscr{Q}}$ constructed in [66].

Remark 16.0.4. In the notation of [66], a module $M \in \mathtt{Mod}(\tilde{A}^{(n)})$ corresponds to the representation of the quiver $Q^{(n)}$ which is given by a collection of vector spaces and maps $M(0) = A$, $M(i) = B_i$, $M(a_i) = q_i$, $M(a_i^*) = p_i$. The functors $(\sigma_i^{-1})^*$ on modules correspond to the operations denoted by T_i. For example, $((\sigma_i^{-1})^* M)(a_j)$ means $T_i(q_j)$ in the notation of [66] .

Next, we introduce a slight modification of the GMV action. Let $\overline{Q}_n$ denote the following quiver

$$\overline{Q}_n \;:=\; \begin{tikzcd} 1 \arrow[rr, bend left=40, "a_1"] \arrow[rr, phantom, "\vdots"] \arrow[rr, bend right=20, "a_n"'] & & 0 \arrow[ll, bend right=20, "a_1^*"'] \arrow[ll, bend left=40, "a_n^*"] \end{tikzcd} \tag{16.0.5}$$

Fix an invertible element $\mu \in k^\times$, and define the k-category $A^{(n)}$ by

$$A^{(n)} = k\langle \overline{Q}_n \rangle / (a_i^* a_i = (\mu - 1) e_1)_{i=1,\dots,n} \tag{16.0.6}$$

Notice that the elements $T_i = e_0 + a_i a_i^*$ are invertible in $A^{(n)}$ for all $i = 1, 2, \dots, n$. Hence, formula (13.0.5) still defines a braid group action on $A^{(n)}$.

The k-category $A^{(n)}$ is obtained from $\tilde{A}^{(n)}$ by applying the following two operations:

1. taking the quotient of $\tilde{A}^{(n)}$ modulo the relations $a_i^* a_i = (\mu - 1) e_i$,
2. collapsing the vertices $\{1, \dots, n\}$ into a single vertex 1.

The GMV braid action on $\tilde{A}^{(n)}$ descends to a braid action on $A^{(n)}$, which we will call the *μ-central GMV action*.

One advantage of working with the μ-central GMV action is that it fixes the set of objects of $A^{(n)}$. In particular, one can consider the induced braid action on the endomorphism algebra of any object of $A^{(n)}$. Specifically, let $A^{(n)}(1,1)$

denote the endomorphism algebra of the object '1' in $A^{(n)}$. For $i, j = 1, 2, \ldots, n$, consider the elements $A_{ij} := -a_i^* a_j \in A^{(n)}(1,1)$. Then, it is easy to see that the algebra $A^{(n)}(1,1)$ has the the following presentation

$$A^{(n)}(1,1) = k\langle A_{ij}\rangle/(A_{ii} = 1 - \mu)$$

It is straightforward to compute the induced braid group action on this algebra in terms of the generators A_{ij}. However, we will write the corresponding formulas in terms of other generators a_{ij} related to A_{ij} by a simple rescaling:

$$a_{ij} := \begin{cases} A_{ij}\ , & i < j \\ -\mu^{-1} A_{ij}\ , & i > j \end{cases} \tag{16.0.7}$$

The associative algebra $A^{(n)}(1,1)$ is free on these generators, and the braid group action on $A^{(n)}(1,1)$ is given by

$$\sigma_k\ :\ \begin{cases} a_{ki} \ \mapsto\ a_{k+1,i} - a_{k+1,k}\, a_{ki} & (i \neq k, k+1) \\ a_{ik} \ \mapsto\ a_{i,k+1} - a_{ik}\, a_{k,k+1} & (i \neq k, k+1) \\ a_{k+1,i} \ \mapsto\ a_{ki} & (i \neq k, k+1) \\ a_{i,k+1} \ \mapsto\ a_{ik} & (i \neq k, k+1) \\ a_{k,k+1} \ \mapsto\ -a_{k+1,k} & \\ a_{k+1,k} \ \mapsto\ -a_{k,k+1} & \\ a_{ij} \ \mapsto\ a_{ij} & (i, j \neq k, k+1) \end{cases} \tag{16.0.8}$$

Formulas (16.0.8) first appeared in [84, 85] as a generalization of the classical Magnus action [119]; we therefore call (16.0.8) the *Humphries-Magnus braid action*. The Humphries-Magnus braid action was used by Ng in [120, 121, 122, 123, 124] as part of his definition of the (combinatorial) knot DGA (see, e.g., [124], Definition 3.3). Now, the main results of Part III in relation to Ng's work can be summarized schematically by the following diagram

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\mu\text{-central}\\ \text{GMV action}\end{array}} \arrow[r, "A^{(n)} \mapsto A^{(n)}(1{,}1)"] \arrow[d, "\begin{array}{c}\text{Homotopy}\\ \text{braid closure}\end{array}"'] & \boxed{\begin{array}{c}\text{Humphries-Magnus}\\ \text{action}\end{array}} \arrow[d, "\text{Ng construction}"] \\
\boxed{\text{Knot DG category}} \arrow[r, "\mathcal{A} \mapsto \mathcal{A}(1{,}1)"] & \boxed{\text{Knot DGA}}
\end{tikzcd}
\tag{16.0.9}
$$

CHAPTER 17

THE KNOT DG CATEGORY

Let $\mathcal{C} = \mathtt{dgCat}_k^{\{0,1\}}$ be the category comprising all small DG k-categories with object set $\{0,1\}$. The morphisms of such DG categories in $\mathcal{C}$ are required to be the identity map on the object set $\{0,1\}$. The k-category $A^{(n)}$ defined in (16.0.6) can then be identified with the n-fold coproduct of copies of $A := A^{(1)}$ in the category $\mathcal{C}$. Moreover, the μ-central GMV action is induced by a cocartesian Yang-Baxter operator $\sigma : A \amalg A \to A \amalg A$ given by the same formula as in (13.0.6). The same calculation as in Lemma 16.0.2 shows that this cocartesian Yang-Baxter operator is Reidemeister with torsion given by formula (16.0.3). Moreover, the following lemma gives a σ-natural map (Definition 15.0.1) that can be used to color a knot.

Lemma 17.0.1. *For any element $\lambda \in k^{\times}$, the map $\theta_\lambda : A \to A$ given by $(a, a^*) \mapsto (\lambda^{-1}a, \lambda a^*)$ is σ-natural.*

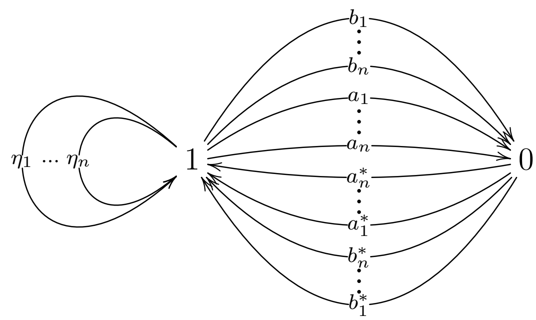
The category $\mathcal{C} = \mathtt{dgCat}_k^{\{0,1\}}$ has a model structure, in which a morphism $f : X \to Y$ is a weak equivalence (resp., fibration) if and only if for any pair of objects $a, b \in X$, the map $f : X(a,b) \to Y(a,b)$ is a quasi-isomorphism (resp., surjection) of chain complexes. This model category is cofibrantly generated, therefore the cofibrations can be characterized as retracts of relative cell complexes (see [77], [11]), which in particular, include semi-free extensions by arrows in non-negative (homological) degree.

One can show that the k-category A viewed as an object of the model category $\mathcal{C} = \mathtt{dgCat}_k^{\{0,1\}}$ is pseudoflat. Therefore, the colored normalized homotopy braid closure (Definition 15.0.2) with respect to the GMV operator (A, σ) and the coloring $\theta = \theta_\lambda$:

$$\begin{aligned} h\bar{\mathcal{L}}_\theta(A,\sigma)[\beta] &:= \operatorname{hocoeq}\Big[\, A^{(n)} \underset{\mathrm{id}}{\overset{\Psi}{\rightrightarrows}} A^{(n)} \,\Big] \\ &= \operatorname{hocolim}\Big[\, A^{(n)} \xleftarrow{(\Psi,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)} \,\Big] \end{aligned} \tag{17.0.2}$$

gives a quasi-isomorphism type in the category $\mathcal{C} = \mathtt{dgCat}_k^{\{0,1\}}$, which is a knot invariant.

We now describe this knot invariant in explicit terms. Let $\mathcal{Q}$ be the following *graded* quiver

$\mathcal{Q} =$ 

where the degrees of arrows are assigned by

$$\deg(a_1) = \ldots = \deg(a_n) = \deg(a_1^*) = \ldots = \deg(a_n^*) = 0$$

$$\deg(b_1) = \ldots = \deg(b_n) = \deg(b_1^*) = \ldots = \deg(b_n^*) = 1$$

$$\deg(\eta_1) = \ldots = \deg(\eta_n) = 2$$

Let $\beta \in B_n$ be a braid that closes to a knot K.

Definition 17.0.3. We defined the *knot DG category of K* to be the DG k-category

$$\mathcal{A}_K = k\langle \mathcal{Q} \rangle / (a_i^* a_i = (\mu - 1)e_1)_{1 \le i \le n}$$

with differential given by

$$\begin{aligned} d(b_i) &= \Psi(a_i) - a_i \\ d(b_i^*) &= \Psi(a_i^*) - a_i^* \\ d(\eta_i) &= -b_i^* a_i - \Psi(a_i^*) b_i \end{aligned} \tag{17.0.4}$$

where $\Psi : A^{(n)} \to A^{(n)}$ is the map defined in (15.0.3).

Our main results regarding the knot DG category $\mathcal{A} = \mathcal{A}_K$ can be encapsulated into the following two theorems.

Theorem 17.0.5. *The knot DG category $\mathcal{A}$ represents the quasi-isomorphism type of the homotopy coequalizer* (17.0.2)*. Hence, the quasi-isomorphism type of the knot DG category $\mathcal{A}$ is independent of the choice of a braid β that closes to a given knot K.*

Theorem 17.0.6. *Let the base commutative ring be $k = \mathbb{Z}[\mu^{\pm 1}, \lambda^{\pm 1}]$. Then, the quasi-isomorphism type of the endomorphism DG algebra $\mathcal{A}(1,1)$ coincides with the quasi-isomorphism type of the knot DGA constructed in [122].*

Theorem 17.0.6 gives an alternative proof of one of the main results in [120, 122] that states that the underlying quasi-isomorphism type of the combinatorial knot DGA is a knot invariant[1].

Proof of Theorem 17.0.6. Define the following morphisms in $\mathcal{A}$, which are elements of the endomorphism DG algebra $\mathcal{A}(1,1)$ of different homological degrees:

$$\begin{aligned} A_{ij} &= -a_i^* a_j \in \mathcal{A}(1,1)_0 \\ B_{ij} &= b_i^* a_j \in \mathcal{A}(1,1)_1 \\ C_{ij} &= a_i^* b_j \in \mathcal{A}(1,1)_1 \\ D_{ij} &= b_i^* b_j \in \mathcal{A}(1,1)_2 \\ e_i &= -\eta_i \in \mathcal{A}(1,1)_2 \end{aligned} \tag{17.0.7}$$

Then, the DG algebra $\mathcal{A}(1,1)$ is freely generated by the elements (17.0.7), modulo the relations $A_{ii} = 1 - \mu$. The differentials of these elements can be easily

[1]Note, however, that the results in *loc. cit.* are slightly stronger as they refer to the invariance of the stable tame isomorphism type rather than the quasi-isomorphism type of the corresponding knot DGA.

computed by the Leibniz rule, using formulas (17.0.4):

$$
\begin{aligned}
d(A_{ij}) &= 0 \\
d(B_{ij}) &= \Psi(a_i^*)a_j - a_i^* a_j \\
d(C_{ij}) &= a_i^*\Psi(a_j) - a_i^* a_j \\
d(D_{ij}) &= (\Psi(a_i^*) - a_i^*)b_j + b_i^*(\Psi(a_j) - a_j) \\
d(e_i) &= b_i^* a_i + \Psi(a_i^*) b_i
\end{aligned}
\tag{17.0.8}
$$

This explicit description allows one to identify $\mathcal{A}(1,1)$ with the combinatorial knot DGA as defined in [122, Definition 2.6] (see also [124]). See [11] for details of this calculation. $\square$

To prove Theorem 17.0.5 one has to calculate the homotopy pushout (17.0.2). As shown in [11], it suffices for this to resolve the right-pointing arrow by a strong cofibration (*i.e.*, a cofibration whose domain is cofibrant), and then take the ordinary pushout of the resulting diagram. Thus, we need to find a semi-free resolution $p : B \overset{\sim}{\twoheadrightarrow} A$ and then construct an appropriate cylinder object $\mathrm{Cyl}(B)$ on B. The right-pointing arrow in the pushout diagram in (17.0.2) will then be resolved by taking the n-fold coproduct $\mathrm{Cyl}(B)^{(n)}$ of this cylinder object.

To construct a semi-free resolution of A, we consider the *graded* quiver

$$
\tilde{Q} \;=\; \left[\begin{tikzcd} 1 \arrow[loop left, "\xi"] \arrow[r, bend left, "a"] & 0 \arrow[l, bend left, "a^*"] \end{tikzcd}\right]
\tag{17.0.9}
$$

where $\deg(a) = \deg(a^*) = 0$ and $\deg(\xi) = 1$. Define $B \in \mathcal{C}$ to be the semi-free DG category $B := k\langle\tilde{Q}\rangle$ with differential given by $d\xi = a^*a - (\mu - 1)e_1$. Then, one can show (see [11]) that the canonical map

$$
p : B \to A, \qquad a \mapsto a, \quad a^* \mapsto a^*, \qquad \xi \mapsto 0
$$

is a quasi-isomorphism. Thus, B can be used as a cofibrant replacement for A.

Next, to define a cylinder on B we will use the construction of canonical cylinder objects for semi-free DG algebras given in [13]. This construction will play an important role in our calculations, so we review it in some detail.

Let R be a DG algebra whose underlying graded algebra is free over a graded k-module V. We write $R = (T(V), d)$. Let $\mathrm{Cyl}(R)$ be the graded algebra defined by $\mathrm{Cyl}(R) := T(V \oplus V' \oplus sV)$, where $sV = V[1]$ is the graded vector space obtained by shifting the (homological) degree of V up by 1. The inclusion of V into the two copies V and V' in $\mathrm{Cyl}(R)$ induces two maps of graded algebra $i : R \to \mathrm{Cyl}(R)$ and $i' : R \to \mathrm{Cyl}(R)$.

We say that a map $S : R \to \mathrm{Cyl}(R)$ of graded k-modules of degree -1 is a (i, i')-*derivation* if, for all homogeneous elements $a, b \in R$, we have

$$S(ab) = S(a) \cdot i'(b) + (-1)^{|a|} i(a) \cdot S(b)$$

It is easy to see that there exists a unique (i, i')-derivation $S : R \to \mathrm{Cyl}(R)$ such that $S(v) = sv \in sV \subset \mathrm{Cyl}(R)$ for all $v \in V$. This derivation S allows us to define a differential on $\mathrm{Cyl}(R)$. Indeed, there exists a unique derivation $d_{\mathrm{Cyl}} : \mathrm{Cyl}(R) \to \mathrm{Cyl}(R)$ of degree -1 satisfying

$$\begin{array}{ll} (1) & d_{\mathrm{Cyl}} \circ i = i \circ d \\ (2) & d_{\mathrm{Cyl}} \circ i' = i' \circ d \\ (3) & d_{\mathrm{Cyl}} \circ S = i - i' - S \circ d \end{array} \tag{17.0.10}$$

It follows from (17.0.10) that $d_{\mathrm{Cyl}}^2 = 0$, which is easy to check on generators of $\mathrm{Cyl}(R)$. Hence, d_{Cyl} makes $\mathrm{Cyl}(R)$ into a DG algebra $\mathrm{Cyl}(R) = (T(V \oplus V' \oplus sV), d_{\mathrm{Cyl}})$.

Next, we define a map $\pi : \mathrm{Cyl}(R) \to R$ by sending the two copies of V in $R = T(V \oplus V' \oplus sV)$ identically onto $V \subset T(V) = R$ and sV to zero. It is straightforward to check that π is a map of DG algebras and, in fact, a quasi-isomorphism from $\mathrm{Cyl}(R)$ onto R. Thus, together with i and i', the map π fits in the diagram $R \amalg R \xhookrightarrow{(i,i')} \mathrm{Cyl}(R) \overset{\pi}{\twoheadrightarrow} R$, which shows that $\mathrm{Cyl}(R)$ is a cylinder object on R. We emphasize that this cylinder object is *canonically* attached to the semi-free DG algebra R. We call it the *Baues-Lemaire cylinder* on R.

The above construction can be naturally extended to semi-free DG categories, *i.e.* DG categories whose underlying graded category is freely generated by a set of arrows. In our present situation, the underlying graded category of the DG category B is freely generated by the graded quiver (17.0.9). Hence, the Baues-Lemaire construction of the cylinder on $R = T(V)$ can be carried over to $B = k\langle\tilde{Q}\rangle$.

Specifically, let $c\tilde{Q} = \tilde{Q} \amalg \tilde{Q}' \amalg (\tilde{Q}[1])$ be the graded quiver

$$c\tilde{Q} \; = \begin{tikzcd} 1 \arrow[loop left, "\xi"] \arrow[loop left, "\xi'"] \arrow[loop left, "\eta"] \arrow[r, bend left=40, "b"] \arrow[r, bend left=20, "a"] \arrow[r, bend left=5, "a'"] & 0 \arrow[l, bend left=5, "a'^*"] \arrow[l, bend left=20, "a^*"] \arrow[l, bend left=40, "b^*"] \end{tikzcd} \tag{17.0.11}$$

which has three copies $\{a, a^*, \xi\}$, $\{a', a'^*, \xi'\}$ and $\{b, b^*, \eta\}$ of the generating arrows of $\tilde{Q}$, with $\{b, b^*, \eta\}$ having homological degree shifted up by 1. Thus,

$$\deg(a) = \deg(a') = \deg(a^*) = \deg(a'^*) = 0$$

$$\deg(\xi) = \deg(\xi') = 1$$

$$\deg(b) = \deg(b^*) = 1$$

$$\deg(\eta) = 2$$

Then, we define $\mathrm{Cyl}(B)$ to be the graded k-category $\mathrm{Cyl}(B) := k\langle c\tilde{Q}\rangle$, with differential $d = d_{\mathrm{Cyl}}$ given by the Baues-Lemaire formulas (17.0.10):

$$\begin{aligned}
d(\xi) &= a^*a - (\mu - 1)e_1 \\
d(\xi') &= a'^*a' - (\mu - 1)e_1 \\
d(b) &= a - a' \\
d(b^*) &= a^* - a'^* \\
d(\eta) &= \xi - \xi' - b^*a' - a^*b
\end{aligned} \tag{17.0.12}$$

For example, by Equation (3) in (17.0.10), we have

$$\begin{aligned}
d(\eta) = d(S(\xi)) &= i(\xi) - i'(\xi) - S(d(\xi)) \\
&= \xi - \xi' - S(a^*a - (\mu - 1)e_1) \\
&= \xi - \xi' - b^*a' - a^*b
\end{aligned}$$

The following proposition implies that $\mathrm{Cyl}(B)$ is indeed a cylinder object on B.

Proposition 17.0.13. *The canonical map* $\pi : \mathrm{Cyl}(B) \to B$ *defined by*

$$\pi(a) = \pi(a') = a, \qquad \pi(a^*) = \pi(a'^*) = a, \qquad \pi(\xi) = \pi(\xi') = \xi,$$
$$\pi(b) = 0, \qquad \pi(b^*) = 0, \qquad \pi(\eta) = 0$$

is a quasi-isomorphism.

Now, as explained in the Introduction, the homotopy pushout (17.0.2) can be computed as the ordinary pushout of the following diagram

$$A^{(n)} \xleftarrow{(\Psi,\mathrm{id}^{(n)})\circ(p^{(n)},p^{(n)})} B^{(n)} \amalg B^{(n)} \xhookrightarrow{(i^{(n)},i'^{(n)})} \mathrm{Cyl}(B)^{(n)} \tag{17.0.14}$$

A straightforward calculation shows that the result is the knot DG category presented in Definition 17.0.3.

CHAPTER 18

THE KNOT CATEGORY

Let K be a knot, and let $\mathcal{A}_K$ be the knot DG category of K presented in Definition 17.0.3.

Definition 18.0.1. We call the 0-th homology of $\mathcal{A}_K$ the *knot k-category* of K and denote it by $A_K := H_0(\mathcal{A}_K)$. This is a k-category whose isomorphism class is a knot invariant.

Let $\pi = \pi_1(\mathbb{R}^3 \backslash K)$ be the knot group of K. Consider the group algebra $k[\pi]$ as a k-category with one object 0. Similarly, the ring k can itself be considered a k-category with one object 1, which we denote by $\mathbf{1}_{\{1\}}$. Let $k[\pi]^+ = k[\pi] \amalg \mathbf{1}_{\{1\}}$ be the disjoint union of these two k-categories. Thus, $k[\pi]^+ \in \mathtt{Cat}_k^{\{0,1\}}$ is a k-category with object set $\{0,1\}$. Now, let $k[\pi]^+\langle a, a^*\rangle$ be the free extension in $\mathtt{Cat}_k^{\{0,1\}}$ of $k[\pi]^+$ by the arrows a, a^* where a goes from the vertex 1 to the vertex 0, while a^* goes in the opposite direction, i.e., from 0 to 1. We will denote this k-category schematically by

$$k[\pi]^+\langle a, a^*\rangle \;=\; \left[\begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \boxed{k[\pi]} \arrow[l, bend left, "a^*"] \end{tikzcd}\right]$$

Then, we have the following description of the knot k-category in terms of the peripheral pair $(\pi, (m, l))$, where $m, l \in \pi$ are respectively a meridian and a longitude of the knot K.

Theorem 18.0.2. *The knot k-category A_K can be described as*

$$A_K \cong k[\pi]^+\langle a, a^*\rangle / J$$

where J is the ideal generated by the following elements

(1) $aa^* + e_0 - m$

(2) $a^*a + e_1 - \mu e_1$

(3) $\lambda a - la\ ,\ \lambda a^* - a^* l\ .$

Remark 18.0.3. A peripheral pair (m, l) is well-defined up to inner automorphisms of π. Suppose that $(m', l') = (\gamma m \gamma^{-1}, \gamma l \gamma^{-1})$ is another such pair, then letting $a' = \gamma a$ and $(a^*)' = a^* \gamma^{-1}$, we reduce the defining relations (1)-(3) of the k-category A_K to the same form written in terms of $a', (a^*)', l', m'$. Hence, up to isomorphism, this k-category is independent of the choice of the peripheral pair.

To prove Theorem 18.0.2 we notice that the braid group B_n acts on the elements $T_i \in A^{(n)}(0,0)$ the same way as it acts on the generators $x_i \in \mathbb{F}_n$ in the Artin representation. This implies that, after taking the categorical braid closure, there is a map ϕ from $k[\pi]$ to the endomorphism algebra $A_K(0,0)$ of the knot k-category at 0, taking x_i to T_i. Define $\tilde{\phi} : k[\pi]^+\langle a, a^* \rangle \to A_K$ by extending the map ϕ, so that $a \mapsto a_1$, and $a^* \mapsto a_1^*$. Then, one can show that if $m = T_1 \in \pi$ and $l \in \pi$ is the corresponding longitude, then the map $\tilde{\phi}$ sends the ideal J defined in Theorem 18.0.2 to zero, and hence descends to a map from the quotient $k[\pi]^+\langle a, a^* \rangle / J$ to A_K, which can be shown to be an isomorphism. (See [11] for details.)

Recall (see Theorem 17.0.6) that the endomorphism DG algebra of the object 1 in the knot DG category is quasi-isomorphic to the knot DGA. Therefore, in particular, the endomorphism algebra of the object 1 of the knot k-category recovers the 0th homology of the knot DGA. Thus, Theorem 18.0.2 implies

Theorem 18.0.4 ([122])**.** *The 0th homology of the knot DGA is isomorphic to the tensor algebra over k freely generated by elements $[\gamma]$, where $\gamma \in \pi_1(\mathbb{R}^3 \setminus L)$, modulo the* *relations*

(1) $[e] = 1 - \mu$, *where* e *is the identity element;*

(2) $[\gamma_1\gamma_2] - [\gamma_1 m\gamma_2] - [\gamma_1][\gamma_2] = 0$ *for* $\gamma \in \pi_1(\mathbb{R}^3 \setminus L)$*;*

(3) $[\gamma l] = [l\gamma] = \lambda[\gamma]$ *for* $\gamma_1, \gamma_2 \in \pi_1(\mathbb{R}^3 \setminus L)$*.*

Proof. The endomorphism algebra of the object 1 of the k-category $k[\pi]^+\langle a, a^*\rangle$ is freely generated by the elements $[\gamma] := -a\gamma a^*$. The ideal J of Theorem 18.0.2 defines relations in this endomorphism algebra, which are simply the three relations given in the theorem. $\square$

Remark 18.0.5. Theorem 18.0.2 also shows that the endomorphism algebra of the knot category at the vertex 0 is given by $A_K(0,0) = k[\pi]/\langle (m-1)(m-\mu)\,,\ (m-1)(l-\lambda)\rangle\,.$

CHAPTER 19

THE FULLY NONCOMMUTATIVE LINK DG CATEGORY

Recall, in Section 16, we have 'simplified' the GMV k-category $\tilde{A}^{(n)}$ by performing the following two operations on the underlying quiver:

1. we have collapsed the vertices $1, \ldots, n$ to a single vertex 1,
2. we have set the elements $e_i + a_i^* a_i$ to be equal to a central element $\mu \in k^\times$.

In this section, we will work with the original GMV category $\tilde{A}^{(n)}$ and the associated braid action. We will show that the corresponding homotopy braid closure is related to the "fully noncommutative knot DGA" introduced in [54], [124].

Let $\tilde{A}^{(n)}$ be the k-category (13.0.4) with the GMV braid action defined as in (13.0.5). Consider the elements $\mu_i = a_i^* a_i + e_i \in \tilde{A}^{(n)}(i,i)$, which are now no longer central.

Suppose we are given a braid $\beta \in B_n$ which closes to a link L with r component $L = L_1 \cup \ldots \cup L_r$. For each $1 \le i \le r$, let S_i be the set of strands $S_i \subset \{1, \ldots, n\}$ that closes to the component L_i. Note that these are precisely the orbits of the cyclic group generated by β acting on the set $\{1, \ldots, n\}$ by permutations.

Now, for each $1 \le i \le r$, identify all the vertices j in $A^{(n)}$ that are in the set S_i to a single vertex i, and identify all the elements μ_j, for $j \in S_i$, to a single element μ_i. Let $\overline{A^{(n)}}$ be the resulting k-category. Then, the action map $\beta : A^{(n)} \to A^{(n)}$ induces $\overline{\beta} : \overline{A^{(n)}} \to \overline{A^{(n)}}$. We can use this induced braid action to define the fully noncommutative link DG category.

Definition 19.0.1. The *fully noncommutative link DG category* of L is the DG category $\tilde{\mathcal{A}}_L$, whose underlying graded k-cateogry is defined to be the quotient of

$$
\begin{tikzcd}
\boxed{k[\lambda_1^{\pm 1}, \mu_1^{\pm 1}]} \arrow[loop left, "\{\eta_j\}_{j\in S_1}"] \arrow[rdd, bend left=40, "\{b_j\}_{j\in S_1}"] \arrow[rdd, bend left=20, "\{a_j\}_{j\in S_1}"] & \\
\vdots & \\
& \bullet \arrow[luu, bend right=10, "\{b_j^*\}_{j\in S_1}"'] \arrow[luu, "\{a_j^*\}_{j\in S_1}"'] \arrow[ldd, bend left=10, "\{b_j^*\}_{j\in S_r}"] \arrow[ldd, bend left=30, "\{a_j^*\}_{j\in S_r}"] \\
\vdots & \\
\boxed{k[\lambda_r^{\pm 1}, \mu_r^{\pm 1}]} \arrow[loop left, "\{\eta_j\}_{j\in S_r}"] \arrow[ruu, "\{b_j\}_{j\in S_r}"] \arrow[ruu, bend right=10, "\{a_j\}_{j\in S_r}"'] &
\end{tikzcd}
\tag{19.0.2}
$$

modulo the relations $e_i + a_j^* a_j = \mu_i$ for all $j \in S_i$, $1 \leq i \leq r$, where the degrees of the generators are given by

$$
\deg(a_j) = \deg(a_j^*) = 0, \qquad \deg(b_j) = \deg(b_j^*) = 1, \qquad \deg(\eta_j) = 2
$$

To define the differential, choose a strand $j_i \in S_i$, one for each $1 \leq i \leq r$, and for each $j \in S_i$, set

$$
\begin{aligned}
d(b_j) &= \begin{cases} \overline{\beta}(a_j)\lambda_i^{-1}\mu_i^{-w_i} - a_j & (j = j_i) \\ \overline{\beta}(a_j) - a_j & (j \neq j_i) \end{cases} \\
d(b_j^*) &= \begin{cases} \lambda_i \mu_i^{w_i}\overline{\beta}(a_j^*) - a_j & (j = j_i) \\ \overline{\beta}(a_j^*) - a_j & (j \neq j_i) \end{cases} \\
d(\eta_j) &= \begin{cases} -b_j^* a_j - \lambda_i \mu_i^{w_i}\overline{\beta}(a_j^*) b_j & (j = j_i) \\ -b_j^* a_j - \overline{\beta}(a_j^*) b_j & (j \neq j_i) \end{cases}
\end{aligned}
\tag{19.0.3}
$$

Theorem 19.0.4. *Let R_0 be the k-category with r objects, given by the disjoint union of k-algebras*

$$
R_0 = k[\lambda_1^{\pm 1}, \mu_1^{\pm 1}] \amalg \ldots \amalg k[\lambda_r^{\pm 1}, \mu_r^{\pm 1}]
$$

Then, the quasi-isomorphism type of the pair $(R_0, \tilde{\mathcal{A}})$ consisting of the k-category R_0, together with the canonical map from R_0 to the fully noncommutative link DG category

$\tilde{\mathcal{A}}$, is a link invariant. Moreover, if we collapse the objects $\{1, \ldots, r\}$ to a single object 1, then the endomorphism DG algebra at this collapsed vertex coincides with the fully noncommutative knot DGA constructed in [54]. (*Here, we take the base commutative ring k to be $\mathbb{Z}$.*)

The first part of the above theorem is proved by interpreting the fully noncommutative link DG category as a homotopy braid closure in a suitable model category. The second part follows from the first by a direct calculation similar to the one in Section 17 (see also the beginning of Section 20). The identification of the fully noncommutative link DG category with a homotopy braid closure is completely parallel to the μ-central case discussed above. The crucial difference, however, is that one should work in a different model category (see [11] for details).

The above theorem identifies the *quasi-isomorphism* type of the pair $(R_0, \tilde{\mathcal{A}})$; however, if we are only interested in the underlying *quasi-equivalence* type, then we have the following

Theorem 19.0.5. *The quasi-equivalence type of the link DG category $\tilde{\mathcal{A}}$ is given by the (normalized) homotopy closure of the braid $\beta \in B_n$ with respect to the GMV operator, taken in the category $\mathtt{dgCat}_k^*$ of pointed DG categories with model structure defined in [152].*

Notice that, in this theorem, no coloring is needed. The extra parameters λ_i are formed in the process of taking the homotopy braid closure. This is not "visible" if we, like in the μ-central case, work with a more rigid model structure, where the weak equivalences are quasi-isomorphisms (*cf.* also Remark 19.0.9). "Normalizing" is also not necessary in Theorem 19.0.5, as it only changes the parameter $\lambda_i \mapsto \lambda_i \, \mu_i^{w_i}$ in R_0.

Definition 19.0.6. The *fully noncommutative link k-category* of a link L is defined to be $\tilde{A}_L := H_0(\tilde{\mathcal{A}}_L)$, the 0th homology of the fully noncommutative link DG category of L.

The k-category $\tilde{A}_L$ can be expressed in terms of the link group, together with meridians and longitudes chosen in each link component. To be precise, let $M = \mathbb{R}^3 \setminus L$ be the link complement. For $1 \leq i \leq r$, let $\partial_i M \subset M$ denote the torus boundary of M corresponding to the link component L_i. Choose base-points $p_i \in \partial_i M$, and $p_0 \in M$. Then, there are canonical meridian and longitude elements $\mu_i, \lambda_i \in \pi_1(\partial_i M, p_i)$, which identifies the group algebra $k[\pi_1(\partial_i M, p_i)]$ as $k[\lambda_i^{\pm 1}, \mu_i^{\pm 1}]$. By choosing a path a_i in M from p_i to p_0, one can define a map $\phi_i : \pi_1(\partial_i M, p_i) \to \pi_1(M, p_0)$. Let m_i and l_i be the images of μ_i and λ_i under ϕ_i, respectively. Then, we have the following description of the fully noncommutative link category.

Theorem 19.0.7. *The fully noncommutative link k-category $\tilde{A}_L$ is the quotient of the k-category*

$$\begin{tikzcd}
\boxed{k[\lambda_1^{\pm 1}, \mu_1^{\pm 1}]} \arrow[drr, bend left, "a_1"] & & \\
\vdots & & \boxed{k[\pi_1(M, p_0)]} \arrow[ull, "a_1^*"] \arrow[dll, bend left, "a_r^*"] \\
\boxed{k[\lambda_r^{\pm 1}, \mu_r^{\pm 1}]} \arrow[urr, "a_r"] & &
\end{tikzcd} \tag{19.0.8}$$

modulo the ideal of relations

(1) $a_i a_i^* + e_0 - m_i$

(2) $a_i^* a_i + e_i - \mu_i$

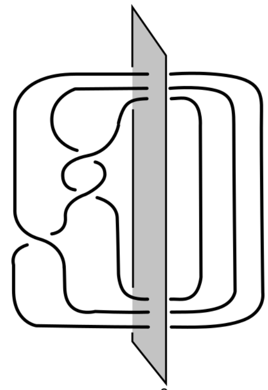
(3) $a_i\lambda_i - l_i a_i\ ,\ \lambda_i a_i^* - a_i^* l_i$

Remark 19.0.9. While the fully noncommutative link DG category is the homotopy braid closure of the GMV action, it is *not* true that its 0-th homology, *i.e.* the fully noncommutative link k-category, is the categorical braid closure of the GMV action. The categorical braid closure can be obtained as a specialization of the fully noncommutative link k-category when all parameters λ_i are set to be 1. This discrepancy is due to the fact that, in Tabuada's model structure on $\mathtt{dgCat}_k$, the weak equivalences are quasi-equivalences, which, by definition, induce equivalences (not isomorphisms) of k-categories at the level of 0-th homology. Then, the homotopy colimits of diagrams in $\mathtt{dgCat}_k$ with respect to Tabuada's model structure induce, at the level of 0th homology, not strict colimits, but rather 2-colimits, which can be viewed, in part, as homotopy colimits. Thus, the fully noncommutative link k-category is already a homotopy braid closure, rather than a strict categorical braid closure.

As mentioned in the introduction, the fully noncommutative link k-category is closely related to perverse sheaves. To be precise, let $\mathcal{S}$ be the stratification on $\mathbb{R}^3$ with two strata $(L, \mathbb{R}^3 \setminus L)$, where L is a link in $\mathbb{R}^3$. Following the degree conventions of [90], we let p be the perversity of $\mathcal{S}$ given by $p(1) = 0$ and $p(3) = -1$. (The values at other integers do not matter.) Then, we have

Theorem 19.0.10. *Suppose that k is a field. The category $\mathsf{Perv}^p(\mathbb{R}^3, L)$ of p-perverse sheaves of k-vector spaces on $\mathbb{R}^3$ constructible with respect to the stratification $\mathcal{S}$ is equivalent to the category of finite-dimensional left modules over the fully noncommutative link category $\tilde{A}_L$.*

Sketch of proof. Suppose that a braid $\beta \in B_n$ is placed in the region $\{x < 0\}$, and closes to the link L by letting the two ends of the braid pass through the hyperplane $\{x = 0\}$ and close in the region $\{x > 0\}$, as in the following diagram.



Let U, V be open subsets of $\mathbb{R}^3$ defined by $U = \{x < \varepsilon\}$ and $V = \{x > -\varepsilon\}$ for some small $\varepsilon > 0$. Then, both the pairs $(U, U \cap L)$ and $(V, V \cap L)$ are diffeomorphic to the pair $(\mathring{D} \times \mathring{I}, \{p_1, \ldots, p_n\} \times \mathring{I})$, where $\mathring{D}$ denotes the interior of the disk and $\mathring{I}$ denotes the open interval $(0, 1)$. The pair $(U \cap V, U \cap V \cap L)$ is diffeomorphic to the pair $(\mathring{D} \times \mathring{I}, \{p_1, \ldots, p_n, p'_1, \ldots, p'_n\} \times \mathring{I})$. Therefore, the category $\mathsf{Perv}^p(U, U \cap L)$ can be identified with the category $\mathsf{Perv}(D, \{p_1, \ldots, p_n\})$ with middle perversity, which, under a suitable choice of 'cuts', is equivalent to the category $\mathtt{Mod}(\tilde{A}^{(n)})$ of finite-dimensional modules over the k-category $\tilde{A}^{(n)}$.

Similar statements are true for the pairs $(V, V \cap L)$ and $(U \cap V, U \cap V \cap L)$. One can show then that the following diagram of restriction functors

$$\mathsf{Perv}^p(U, U \cap L) \to \mathsf{Perv}^p(U \cap V, U \cap V \cap L) \leftarrow \mathsf{Perv}^p(V, V \cap L) \qquad (19.0.11)$$

is equivalent to the following diagram of functors

$$\mathtt{Mod}(A^{(n)}) \xrightarrow{(\beta^*, \mathrm{id})} \mathtt{Mod}(A^{(2n)}) \xleftarrow{(\mathrm{id}, \mathrm{id})} \mathtt{Mod}(A^{(n)}) \qquad (19.0.12)$$

Since perverse sheaves form a stack (see [90, Propositions 10.2.7 and 10.2.9]), the category $\mathsf{Perv}^p(\mathbb{R}^3, L)$ is equivalent to the 2-limit of the diagram (19.0.11), and hence of the diagram (19.0.12). This implies the desired result. For details, see [11]. $\square$

When combined with Theorem 19.0.7, Theorem 19.0.10 gives a description of the category $\mathsf{Perv}(\mathbb{R}^3, L)$ of perverse sheaves in terms of linear algebra data, similar in spirit to the original description of the category $\mathsf{Perv}(D, \{p_1, p_2, \ldots, p_n\})$ given in [66].

CHAPTER 20

GENERALIZATIONS AND FURTHER QUESTIONS

In the GMV braid action, the group B_n acts on the generators $T_i = a_i a_i^* + e_0$ via the Artin representation (13.0.1). Thus, regarding the free group $\mathbb{F}_n$ as a category with a single object, we can regard the GMV action as an extension of the Artin action. In [165], Wada constructed several examples of braid group actions on $\mathbb{F}_n$ generalizing the classical Artin representation. Like the Artin representation, Wada's braid group actions are local and homogeneous, *i.e.* generated by a single cocartensian Yang-Baxter operator on $\mathbb{F}_1$. It is therefore natural to ask whether they admit extensions similar to the GMV extension.

Consider, for example, the following cocartesian Yang-Baxter operator constructed in [165]:

$$\sigma : \mathbb{F}_2 \to \mathbb{F}_2 \qquad x_1 \mapsto x_1^N x_2 x_1^{-N}, \qquad x_2 \mapsto x_1 \, ,$$

where N is an arbitrary (fixed) integer.

This action does admit an extension similar to the GMV action. Indeed,

$$\sigma_k \ : \ \begin{cases} a_i \ \mapsto \ a_i & (i \neq k, k+1) \\ a_k \ \mapsto \ T_k^N a_{k+1} & \\ a_{k+1} \ \mapsto \ a_k & \\ a_i^* \ \mapsto \ a_i^* & (i \neq k, k+1) \\ a_k^* \ \mapsto \ a_{k+1}^* T_k^{-N} & \\ a_{k+1}^* \ \mapsto \ a_k^* & \end{cases} \tag{20.0.1}$$

Note that, for $N = 1$, this is the original GMV action (13.0.5). Moreover, using a result of [44], one can show that the actions (20.0.1) are non-equivalent to each other for different N's; thus, for $N \neq 1$, (20.0.1) is a genuine generalization of the GMV action.

The elements $T_i^{\pm N}$ can be written in an alternative form involving the conjugate elements $\mu_i = e_i + a_i^* a_i \in \tilde{A}^{(n)}(i,i)$. (We recall that μ_i are no longer central elements in $\tilde{A}^{(n)}(i,i)$.). Indeed, by an induction, one can show that

$$T_i^N = e_0 + a_i\,[N]_{\mu_i}\,a_i^* \qquad \text{for all } N \in \mathbb{Z}$$

where $[N]_\mu \in k$ are the "quantum integers" defined by

$$[N]_{\mu_i} = \frac{\mu_i^N - 1}{\mu_i - 1} := \begin{cases} e_i + \mu_i + \mu_i^2 + \ldots + \mu_i^{N-1} & \text{if } N > 0 \\ 0 & \text{if } N = 0 \\ -\mu_i^{-1} - \mu_i^{-2} - \ldots - \mu_i^{N} & \text{if } N < 0 \end{cases}$$

As in Section 16, we set $A_{ij} = -a_i^* a_j$ for all i, j. Then, we have the following formulas defining the braid group action on the restriction of the k-category $\tilde{A}$ to the vertices $\{1, \ldots, r\}$:

$$\sigma_k \;:\; \begin{cases} A_{ki} \;\mapsto\; A_{k+1,i} - A_{k+1,k}\,[-N]_{\mu_k}\,A_{ki} & i \neq k, k+1 \\ A_{ik} \;\mapsto\; A_{i,k+1} - A_{ik}\,[N]_{\mu_k}\,A_{k,k+1} & i \neq k, k+1 \\ A_{k+1,i} \;\mapsto\; A_{ki} & i \neq k, k+1 \\ A_{i,k+1} \;\mapsto\; A_{ik} & i \neq k, k+1 \\ A_{k,k+1} \;\mapsto\; A_{k+1,k}\,\mu_k^{-N} & \\ A_{k+1,k} \;\mapsto\; \mu_k^N\,A_{k,k+1} & \\ A_{ij} \;\mapsto\; A_{ij} & i, j \neq k, k+1 \\ \mu_k \;\mapsto\; \mu_{k+1} & \\ \mu_{k+1} \;\mapsto\; \mu_k & \\ \mu_i \;\mapsto\; \mu_i & i \neq k, k+1 \end{cases}$$

Now, for $i \neq j$, define

$$a_{ij} = \begin{cases} A_{ij}\,[N]_{\mu_j}, & i < j \\ A_{ij}\,[-N]_{\mu_j}, & i > j \end{cases} \tag{20.0.2}$$

Then, the above action becomes

$$\sigma_k \; : \; \begin{cases} a_{ki} \;\mapsto\; a_{k+1,i} - a_{k+1,k}\, a_{ki} & i \neq k, k+1 \\ a_{ik} \;\mapsto\; a_{i,k+1} - a_{ik}\, a_{k,k+1} & i < k \\ a_{ik} \;\mapsto\; a_{i,k+1} - a_{ik}\, \mu_k^N\, a_{k,k+1}\, \mu_{k+1}^{-N} & i > k+1 \\ a_{k+1,i} \;\mapsto\; a_{ki} & i \neq k, k+1 \\ a_{i,k+1} \;\mapsto\; a_{ik} & i \neq k, k+1 \\ a_{k,k+1} \;\mapsto\; -a_{k+1,k} & \\ a_{k+1,k} \;\mapsto\; -\mu_k^N\, a_{k,k+1}\, \mu_{k+1}^{-N} & \\ a_{ij} \;\mapsto\; a_{ij} & i, j \neq k, k+1 \\ \mu_k \;\mapsto\; \mu_{k+1} & \\ \mu_{k+1} \;\mapsto\; \mu_k & \\ \mu_i \;\mapsto\; \mu_i & i \neq k, k+1 \end{cases}$$

For $N = 1$, this coincides with the 'fully noncommutative' action defined in [54] (see also [124, Appendix]).

In a different direction, one can also construct a large family of GMV-type braid actions by extending the family of generalized Artin actions found in [44]. Specifically, let $B \in \mathtt{Alg}_k$ be an associative algebra over k, and let $x, y \in B^\times$ be a pair of *invertible* and *commuting* elements. Let $\hat{A}^{(n)}$ be the k-category given by

$$\hat{A}^{(n)} = \left[\begin{tikzcd} \bullet \arrow[dr, bend left, "a_1"] & \cdots & \bullet \arrow[dl, bend left, "a_n"] \\ & \boxed{B * \overset{n}{\cdots} * B} \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_n^*"] & \end{tikzcd} \right]$$

which can be interpreted as an n-fold coproduct in the category $\mathtt{Cat}_k^*$ of (small) pointed k-categories. Then, one can check by a direct calculation that the following assignments define a braid group action on $\hat{A}^{(n)}$.

$$\sigma_k \;:\; \begin{cases} a_k \;\mapsto\; x_k a_{k+1} \\ a_k^* \;\mapsto\; a_{k+1}^* x_k^{-1} \\ b_k \;\mapsto\; x_k b_{k+1} x_k^{-1} \\ a_{k+1} \;\mapsto\; y_k a_k \\ a_{k+1}^* \;\mapsto\; a_k^* y_k^{-1} \\ b_{k+1} \;\mapsto\; y_k b_k y_k^{-1} \\ a_i \;\mapsto\; a_i & (i \neq k, k+1) \\ a_i^* \;\mapsto\; a_i^* & (i \neq k, k+1) \\ b_i \;\mapsto\; b_i & (i \neq k, k+1) \end{cases} \tag{20.0.3}$$

where b_i is the element $b \in B$ put in the i-th copy of B in $B^{(n)} := B * \overset{n}{\ldots} * B$. Notice that, when $B = k[H]$ is the group algebra of a group H, and when $x = h \in H$ and $y = h^{-1}$, the braid action on $H^{(n)} \subset k[H]^{(n)}$ at the vertex 0 coincides with the action defined in [44].

Consider any ideal $I \subset \hat{A}(0,0) = B\langle aa^* \rangle$, and let $\hat{A}/I$ be the k-category obtained by quotienting $\hat{A}$ by the ideal generated by I. Then, for any element $f \in I$, we have

$$\sigma_k(f_k) = x_k f_{k+1} x_k^{-1} \qquad \sigma_k(f_{k+1}) = y_k f_k y_k^{-1} \qquad \sigma_k(f_i = f_i \text{ if } i \neq k, k+1)$$

Therefore, the cocartesian Yang-Baxter operator corresponding to the above braid group action descends to the quotient $\hat{A}/I$. If we take $B = k[T^{\pm}]$, $x = T$ and $y = 1$, and consider the ideal I generated by the element $aa^* + 1 - T$, then the resulting quotient $\hat{A}/I$, together with its corresponding cocartesian Yang-Baxter operator σ, is equivalent to the k-category $\tilde{A}$, together with the GMV operator, constructed in Section 16.

Theorem 20.0.4. *The braid group actions* (20.0.1) *and* (20.0.3) *are generated by Reidemeister operators in the category* $\mathtt{dgCat}_k^*$ *of pointed DG categories on objects* $\tilde{A}$ *and* $\hat{A}/I$*, respectively. These objects are pseudoflat with respect to Tabuada's model structure on* $\mathtt{dgCat}_k^*$*. Thus, the homotopy braid closure with respect to these operators gives link invariants that generalize the fully noncommutative link DG category* $\tilde{\mathscr{A}}_L$*.*

We conclude Part III with a few questions and remarks.

1. *Constructible sheaves and contact homology.* Recently, some interesting work has been done on the geometric side of contact homology relating it to constructible sheaves (see [56, 130, 150, 143, 144]). It would be interesting to understand our Theorem 19.0.10 in this geometric context and more generally, to clarify the meaning of our construction from Floer-theoretic and constuctible sheaves point of view.

In more detail, the relation between Legendrian contact homology and constructible sheaves is based on a theorem of Nadler and Zaslow [125] (see also [126]) that, for any real analytic manifold M, establishes an equivalence between the derived category $D_c(M)$ of constructible sheaves on M and the derived Fukaya category $D\mathsf{Fuk}(T^*M)$ of the cotangent bundle T^*M of M. This equivalence of triangulated categories is induced by a quasi-equivalence of A_∞-categories $\mu : \mathsf{Sh}_c(M) \to \mathrm{Tw}\mathsf{Fuk}(T^*M)$, where $\mathsf{Sh}_c(M)$ is a DG category defined as the DG quotient of the (naive) DG category of constructible sheaves on M modulo acyclic complexes and $\mathrm{Tw}\mathsf{Fuk}(T^*M)$ is the A_∞-category of twisted complexes in the Fukaya category $\mathsf{Fuk}(T^*M)$. The functor μ can be viewed as a categorification of the classical characteristic cycle construction and is called the *microlocalization functor*.

Now, for any conical Lagrangian submanifold $\tilde{\Lambda} \subseteq T^*M$, the restriction of the microlocalization functor to the subcategory $\mathsf{Sh}_c(M)_{\tilde{\Lambda}} \subseteq \mathsf{Sh}_c(M)$ of constructible sheaves with singular support in $\tilde{\Lambda}$ gives a quasi-equivalence $\mu : \mathsf{Sh}_c(M)_{\tilde{\Lambda}} \xrightarrow{\sim} \mathrm{Tw}\mathsf{Fuk}(T^*M)_{\tilde{\Lambda}}$ onto the full subcategory $\mathrm{Tw}\mathsf{Fuk}(T^*M)_{\tilde{\Lambda}}$ of the twisted Fukaya category consisting of Lagrangians whose boundary at infinity lies in the boundary of $\tilde{\Lambda}$. Such a submanifold $\tilde{\Lambda}$ is determined by its intersection $\Lambda := \tilde{\Lambda} \cap ST^*M$ with the unit cotangent bundle of M; the bundle ST^*M has a natural contact structure, and Λ is a Legendrian submanifold of ST^*M. It turns out that the Legendrian contact homology (LHC) of the pair (ST^*M, Λ) is related to the Fukaya category $\mathrm{Tw}\mathsf{Fuk}(T^*M)_{\tilde{\Lambda}}$ and hence, via the microlocalization functor, to the sheaf category $\mathsf{Sh}_c(M)_{\tilde{\Lambda}}$. More precisely, it is expected that the complexes of constructible sheaves in $\mathsf{Sh}_c(M)_{\tilde{\Lambda}}$ determine augmentations of the Legendrian DGA of (ST^*M, Λ) via a geometric symplectic filling construction.

In the case of one-dimensional Legendrians, this relation has been worked out in detail in [130, 150]. Specifically, if $M = \mathbb{R}^2$, then $ST^*\mathbb{R}^2 \cong \mathbb{R}^2 \times S^1$ contains an open contact submanifold $\mathbb{R}^3 \subset \mathbb{R}^2 \times S^1$. Hence, any Legendrian link $L \subset \mathbb{R}^3$ can be considered as a Legendrian submanifold in $ST^*\mathbb{R}^2$. In [130], for a Legendrian link $L \subset \mathbb{R}^3$, the authors construct a (unital) A_∞-category $\mathsf{Aug}_+(L)$, whose objects are augmentations of the Chekanov-Eliashberg DG algebra of L, and show that there is an A_∞-equivalence $\mathsf{Aug}_+(L) \simeq \mathcal{C}_1(L)$, where $\mathcal{C}_1(L)$ is the full subcategory of $\mathsf{Sh}_c(\mathbb{R}^2)_{\tilde{L}}$ consisting of sheaves of 'microlocal rank one along the link L'.

A possible extension of this equivalence to higher dimensions (specifically, to the case of knot contact homology and knot DGA in $\mathbb{R}^3$) has been recently proposed by V. Shende et. al. (see, e.g., [143, Section 4], [56, Section 6.6], [144, Section 6.5]). In this case, $M = \mathbb{R}^3$ and the Legendrian $\Lambda \subset ST^*M$ is given by the unit conormal bundle $\Lambda_L := ST^*_L\mathbb{R}^3$ associated to a link $L \subset \mathbb{R}^3$. It is interesting that the support condition defining the subcategory $\mathsf{Sh}_c(\mathbb{R}^3)_{\tilde{\Lambda}} \subset \mathsf{Sh}_c(\mathbb{R}^3)$ coincides with the constructibility condition in our Theorem 19.0.10, and some geometric arguments suggest that there is a relation between this sheaf category and knot contact homology (see [56, Section 6.6]). Whether this geometric relation can be used to prove the result of Theorem 19.0.10 is not clear to us at the moment: *a priori*, the equivalence of categories in Theorem 19.0.10 originates from a different direction. In fact, there are three approaches to knot contact homology:

1. combinatorial knot contact homology,
2. Legendrian contact homology of the pair $\Lambda_L \subset ST^*\mathbb{R}^3$,
3. constructible sheaves on $\mathbb{R}^3$ with singular support in $\tilde{\Lambda}_L$.

The papers [54, 55] establish an equivalence between (1) and (2) by identifying the generators of the combinatorial knot DGA with Reeb cords and defining the differentials in terms of pseudoholomorphic disks. The geometric approach of [56, 143, 144] relates (2) and (3) via the geometry of symplectic fillings. Our result, Theorem 19.0.10, establishes the relation between (1) and (3) by appealing to the classical description of perverse sheaves on the disk in terms of nearby and vanishing cycle functors [66] and using an algebraic 'gluing' construction (homotopy braid closure). It would be interesting to see whether these approaches actually 'agree'; in particular, can one prove Theorem 19.0.10 using the approach of [56, 143, 144]?

2. *Categorification of the link DG category.* There seems to be a natural way to categorify the DG category $\tilde{\mathcal{A}}$, using the notion of 'perverse schobers' introduced in [102] (see also [103]). First, one can construct a (higher) category $\mathscr{C}$ of $(\infty, 2)$-categories that includes the category $\mathtt{dgCat}_k$ as an object (see [160, 64]). In $\mathscr{C}$, one can find an object $\mathscr{A}^{(n)}$ such that the category of 2-representations of $\mathscr{A}^{(n)}$, *i.e.* an appropriately defined internal hom $\underline{\mathrm{Hom}}_{\mathscr{C}}(\mathscr{A}^{(n)}, \mathtt{dgCat}_k)$, is equivalent to the (higher) category of perverse schobers on the disk with n marked points. Then, there should exist a B_n-action on $\mathscr{A}^{(n)}$ for all $n \geq 1$ that would allow us to take the homotopy braid closure. The result should be an object in $\mathscr{C}$ (*i.e.*, an $(\infty, 2)$-category $\mathbb{A}$), whose category $\underline{\mathrm{Hom}}_{\mathscr{C}}(\mathbb{A}, \mathtt{dgCat}_k)$ of 2-representations is equivalent to a category of 'perverse schobers on $\mathbb{R}^3$ singular along a link'.

3. *Yang-Baxter operators related to coherent sheaves.* Many interesting examples of braid group actions related to coherent sheaves have been constructed in the literature (see, e.g., [149], [142],[8] and references therein). It would be interesting to look at these examples in relation to the examples studied in this Part of the thesis and clarify the relations between the corresponding link invariants.

Part IV

Relative Calabi-Yau completions

CHAPTER 21
INTRODUCTION

In the seminal paper [68], V. Ginzburg introduced the notion of an n-Calabi-Yau differential graded (DG) algebra. For any bimodule M over a DG algebra A, consider its bimodule dual $M^!$ in the derived category of A-bimodule $\mathcal{D}(A^{\mathrm{e}})$ defined as

$$M^! := \boldsymbol{R}\underline{\mathrm{Hom}}_{A^{\mathrm{e}}}(M, A^{\mathrm{e}})$$

A homologically smooth[1] DG algebra A is then said to be *n-Calabi-Yau* if the n-shifted bimodule dual $A^![n] \in \mathcal{D}(A^{\mathrm{e}})$ of A is isomorphic to A itself in the derived category $\mathcal{D}(A^{\mathrm{e}})$. *i.e.*,

$$A^![n] \cong A$$

A class of examples of 3-Calabi-Yau DG algebras, usually called *Ginzburg DG algebras*, was also constructed in [68]. We recall this construction now.

Let A be the free associative algebra $A = k\langle x_1, \dots, x_m \rangle$. M. Kontsevich introduced in [95] a linear map $\frac{\partial}{\partial x_j} : A/[A,A] \to A$ for each $j = 1 \dots, m$, defined by

$$\frac{\partial}{\partial x_j}(x_{i_1} \dots x_{i_r}) := \sum_{\{l \,|\, i_l = j\}} x_{i_l+1} \dots x_{i_r} x_{i_1} \dots x_{i_l - 1}$$

Given any element $\Phi \in A/[A,A]$, called a *potential*, the Ginzburg DG algebra is defined to be the DG algebra

$$\mathscr{D} = k\langle x_1, \dots, x_m, \theta_1, \dots, \theta_m, c \rangle, \quad \deg(c) = 2, \quad \deg(\theta_j) = 1, \quad j = 1, \dots m \tag{21.0.1}$$

[1] A DG algebra A is said to be homologically smooth if it is perfect as a bimodule over itself (see Definition 22.1.8 and 22.1.10). This condition holds if A satisfies some finiteness condition, which we may assume to be automatic for the purpose of this thesis.

with differentials given by

$$d(\theta_j) = \frac{\partial \Phi}{\partial x_j} \qquad \text{and} \qquad d(c) = \sum_{j=1}^{m} [x_j, \theta_j]$$

Ginzburg showed that, if the DG algebra $\mathscr{D}$ is acyclic in positive degree, then it is 3-Calabi-Yau (see [68, Remark 5.3.2]). It was later observed by B. Keller and proved by M. Van den Bergh (see [93]) that Ginzburg DG algebras are always 3-Calabi-Yau.

To see why one should expect this to be true, we consider a general semi-free DG algebra $B = k\langle y_1 \dots y_p \rangle$. To verify the Calabi-Yau property of B, one has to take the derived dual of B as a bimodule over itself. The first step is to find a semi-free resolution[2] of B as a bimodule over itself. To this ends, we consider the following standard resolution of B.

Since B is semi-free, its bimodule of differentials $\Omega^1(B)$ is semi-free with basis $\{Dy_1, \dots, Dy_p\}$, where we have denoted the universal derivation by $D : B \to \Omega^1(B)$. By definition of the bimodule of differentials, one has a short exact sequence

$$0 \to \Omega^1(B) \xrightarrow{\alpha} B \otimes B \xrightarrow{m} B \to 0$$

where $\alpha(Df) = f \otimes 1 - 1 \otimes f$. Thus, the cone of α gives a semi-free bimodule resolution of B. We call this the *standard bimodule resolution* of B, and denote it as

$$\mathrm{Res}(B) := \mathrm{cone}\,[\,\Omega^1(B) \xrightarrow{\alpha} B \otimes B\,] \tag{21.0.2}$$

This semi-free resolution has basis $\{\, E_0\,,\, sDy_1\,, \dots sDy_p \,\}$, where sDy_j refers to the basis element $Dy_j \in \Omega^1(B)$, with degree shifted by 1 by the operator s, and

[2] Strictly speaking, it is not enough to require the resolution to be semi-free. One should also require a Sullivan condition (see Definition 22.1.7 and Remark 22.1.9) in order to guarantee that the resolution is cofibrant.

E_0 refers to the element $1 \otimes 1 \in B \otimes B$.

In particular, if we take B to be the Ginzburg DG algebra (21.0.1), then its standard bimodule resolution $\mathrm{Res}(\mathscr{D})$ is semi-free over the basis

$$\{ \, (E_0)^{(0)} \, , \, (sDx_1)^{(1)} \, , \ldots , \, (sDx_m)^{(1)} \, (sD\theta_1)^{(2)} \, , \ldots , \, (sD\theta_m)^{(2)} \, , \, (sDc)^{(3)} \, \} \quad (21.0.3)$$

where the superscripts indicate the degree of each generator.

Thus, there is a perfect symmetric bimodule pairing

$$\langle - , - \rangle \, : \, \mathrm{Res}(\mathscr{D}) \otimes \mathrm{Res}(\mathscr{D}) \, \to \, \mathscr{D} \otimes \mathscr{D} \quad (21.0.4)$$

of degree -3, which pairs E_0 with sDc, and sDx_i with $sD\theta_i$. If one can show that this pairing is compatible with differentials, then one can show that $\mathscr{D}$ is 3-Calabi-Yau. Van den Bergh showed in [93] that this is indeed the case, and hence proved the 3-Calabi-Yau property of the Ginzburg DG algebra $\mathscr{D}$.

Notice that, two points are important to this proof. The first point is that, the Ginzburg DG algebra was constructed in such a way that the extra generators $\theta_1, \ldots, \theta_m, c$ are added to the algebra $A = \langle x_1, \ldots x_m \rangle$ such that θ_i would produce a dual $sD\theta_i$ to sDx_i in (21.0.3), and c would produce a dual sDc to E_0 in (21.0.3). The second, and more crucial, point is that the differentials of these extra generators $\theta_1, \ldots, \theta_n, c$ are defined in such a way that he pairing (21.0.4) is compatible with differentials.

It is therefore natural to ask whether the definition of the Ginzburg DG algebra can be generalized to give a universal construction that adds generators to a semi-free DG algebra A in a controlled way such that the result is an n-Calabi-Yau algebra. This is indeed possible, and is accomplished by B. Keller in [93]. The construction is called *deformed Calabi-Yau completion*.

Before describing this construction, we pause to remark that, while adic-completion is indeed studied in the literature in relation to Calabi-Yau algebras (see, e.g., [68, 162]), this is not what we refer to when we speak of Calabi-Yau completions. Instead, one should think of Calabi-Yau completion as a noncommutative analogue of completing a vector space V to a shifted symplectic vector space $V \oplus V^*[n]$.

Thus, given any semi-free DG algebra $A = k\langle x_1, \ldots x_m \rangle$, the above discussion suggests that one should form a semi-free DG algebra $\Pi_n(A)$ whose underlying graded algebra is given by

$$\Pi_n(A) := k\langle x_1, \ldots, x_m, \theta_1, \ldots, \theta_m, c\rangle \tag{21.0.5}$$

where the extra generators have degrees $|\theta_j| = n - 2 - |x_i|$, and $|c| = n - 1$.

The graded algebra $\Pi_n(A)$ can be written alternatively as a tensor algebra $\Pi_n(A) = T_A(M)$ of a graded bimodule M over the underlying graded algebra A, where M is free as a bimodule over the basis $\{ c, \theta_1, \ldots, \theta_m \}$. This suggests that one should search for a bimodule universally associated to A with a basis that is in a natural degree-preserving bijection with the set $\{ c, \theta_1, \ldots, \theta_m \}$.

One candidate for such a bimodule is the shifted dual $\mathrm{Res}(A)^\vee[n-1]$ of the standard resolution of A. This bimodule has a basis $\{ s^{n-1}(E_0)^\vee, s^{n-1}(sDx_1)^\vee, \ldots, s^{n-1}(sDx_m)^\vee \}$ dual to the standard basis of $\mathrm{Res}(A)$. The basis elements have degrees $|s^{n-1}(E_0)^\vee| = n - 1 = |c|$ and $|s^{n-1}(sDx_1)^\vee| = n - 2 - |x_i| = |\theta_i|$. Keller took this as the definition of the *n-Calabi-Yau completion*

$$\Pi_n(A) := T_A(\mathrm{Res}(A)^\vee[n-1]) \tag{21.0.6}$$

This definition also specifies the differential on $\Pi_n(A)$.

One can check directly that, if we take A to be concentrated in degree 0, then the 3-Calabi-Yau completion is precisely the Ginzburg DG algebra $\mathscr{D}$, with zero potential $\Phi = 0$. The case when the potential is nonzero then corresponds to a deformation of the 3-Calabi-Yau completion. In fact, Keller showed that every Hochschild homology class $[\eta] \in \mathrm{HH}_{n-2}(A)$ allows one to deform the differentials of the n-Calabi-Yau completion $\Pi_n(A)$. The result is called the *deformed n-Calabi-Yau completion* with respect to the deformation parameter $[\eta] \in \mathrm{HH}_{n-2}(A)$. In the case of the free associative algebra A concentrated in degree 0, any potential $\Phi \in A/[A, A]$ specifies a cyclic homology class $\Phi \in A/[A, A] = \mathrm{HC}_0(A)$. The image $B(\Phi) \in \mathrm{HH}_1(A)$ of this class under the Connes operator $B : \mathrm{HC}_0(A) \to \mathrm{HH}_1(A)$ then gives a Hochschild homology class which serves as a deformation parameter. The resulting deformed 3-Calabi-Yau algebra is precisely the Ginzburg DG algebra $\mathscr{D}$.

As suggested by the name, Keller claimed that the deformed n-Calabi-Yau completion is always n-Calabi-Yau. We will give a counter-example to this claim at the end of Section 23.3. However, we will show that, if the deformation parameter $[\eta] \in \mathrm{HH}_{n-2}(A)$ has a lift to a class in the negative cyclic homology $[\tilde{\eta}] \in \mathrm{HC}^-_{n-2}(A)$, then this claim is indeed true.

We say that a negative cyclic class $[\tilde{\eta}] \in \mathrm{HC}^-_r(A)$ is a lift of a Hochschild class $[\eta] \in \mathrm{HH}_r(A)$ if $[\eta]$ is the image of $[\tilde{\eta}]$ under the canonical map $h : \mathrm{HC}^-_*(A) \to \mathrm{HH}_*(A)$. In this case, we think of $[\tilde{\eta}]$ as an enrichment of $[\eta]$. The above discussion then shows that such enrichments are important in the construction of deformed Calabi-Yau completions. In fact, the importance of negative cyclic enrichment was already suggested in [94] in relation to the very definition of Calabi-Yau algebras.

Recall that, we called a homologically smooth DG algebra n-Calabi-Yau if there is an isomorphism $A^![n] \cong A$ in the derived category $\mathcal{D}(A^{\mathrm{e}})$ of A-bimodules. In fact, since A is assumed to be perfect as a bimodule over itself, we have

$$\boldsymbol{R}\underline{\mathrm{Hom}}_{A^{\mathrm{e}}}(A^![n], A) \simeq A \otimes^{\boldsymbol{L}}_{A^{\mathrm{e}}} A[-n]$$

This shows that maps $[\hat{\xi}] : A^![n] \to A$ in the derived category $\mathcal{D}(A^{\mathrm{e}})$ correspond bijectively to Hochschild classes $[\xi] \in \mathrm{HH}_n(A)$. It was long thought that the Hochschild class that defines the isomorphsim $[\hat{\xi}] : A^![n] \to A$ in an n-Calabi-Yau algebra A should have some enrichment. One such enrichment was suggested by Kontsevich and Vlassopoulos in [94]. They proposed that one should require $[\xi]$ to have a negative cyclic lift. We will adopt this suggestion hereafter, and change our terminology to reflect this choice.

Definition 21.0.7. An *n-Calabi-Yau structure* on a DG algebra A is a negative cyclic class $[\tilde{\xi}] \in \mathrm{HC}^-_n(A)$ whose underlying Hochschild class $[\xi] := h([\tilde{\xi}]) \in \mathrm{HH}_n(A)$ induces an isomorphism $[\hat{\xi}] : A^![n] \xrightarrow{\sim} A$ in the derived category $\mathcal{D}(A^{\mathrm{e}})$ of bimodules.

We will prove the following result (see also Theorem 23.2.12 below for a more detailed specification of finiteness assumptions on A)

Theorem 21.0.8. *Any negative cyclic lift $[\tilde{\eta}] \in \mathrm{HC}^-_{n-2}(A)$ of the deformation parameter $[\eta] \in \mathrm{HH}_{n-2}(A)$ determines a canonical n-Calabi-Yau structure on the deformed n-Calabi-Yau completion $\Pi := \Pi_n(A;\eta)$.*

In view of this theorem, we may view the negative cyclic class $[\tilde{\eta}] \in \mathrm{HC}^-_{n-2}(A)$ as the "true" deformation parameter. This is consistent with the results in [163].

To guide the interested readers, we briefly discuss the proof of this theorem. Details can be found in Section 23.3. To show an isomorphism between $\Pi^![n]$ and Π in the derived category $\mathcal{D}(\Pi^e)$, one could construct a map $\mathrm{Res}(\Pi)^\vee[n] \to \mathrm{Res}(\Pi)$ of Π-bimodules. This corresponds to giving a closed element ξ of degree n in the chain complex

$$\mathbb{X}(\Pi) := \mathrm{Res}(\Pi) \otimes_{\Pi^e} \mathrm{Res}(\Pi)$$

which we call the double X-complex of Π.

However, it is usually very difficult to show that a given element $\xi \in \mathbb{X}(\Pi)$ is closed. The differential in the chain complex $\mathbb{X}(\Pi)$ depends on both the differential on the DG algebra A, as well as on the deformation parameter η, in a subtle way. To overcome this difficulty, we construct in Section 22.5 a map $B : X(\Pi) \to \mathbb{X}(\Pi)$ from the X-complex $X(\Pi) := \Pi \otimes_{\Pi^e} \mathrm{Res}(\Pi)$ to the double X-complex $\mathbb{X}(\Pi)$ of Π. This map lifts the Connes operator $B : X(\Pi) \to X(\Pi)$, and is therefore called the *lifted Connes operator*.

Using the lifted Connes operator, one can construct a *closed* element $\xi \in \mathbb{X}(\Pi)$ that induces an isomorphism $\hat{\xi} : \mathrm{Res}(\Pi)^\vee[n] \xrightarrow{\cong} \mathrm{Res}(\Pi)$. Moreover, by construction, this element will have a natural negative cyclic lift. This completes the proof of Theorem 21.0.8.

So far, we have been dealing with DG algebras. In fact, the universal construction of deformed Calabi-Yau completions admits direct generalizations from DG algebras to DG categories. The notion of semi-free DG algebras is then replaced by the notion of semi-free DG categories. These are DG categories $\mathcal{A}$ whose underlying graded k-categories are freely generated by a graded quiver Q over an object set R. We write this as $\mathcal{A} = T_R(Q)$. In forming the deformed Calabi-Yau completion of a semi-free DG category $\mathcal{A} = T_R(Q)$, we add generating arrows $f^\vee$ in direction opposite to those in Q, as well as a loop generator c_x for each $x \in R$. Thus, we have

$$\Pi_n(\mathcal{A};\eta) \;=\; T_R(\,\{f\}_{f\in Q} \cup \{f^\vee\}_{f\in Q} \cup \{c_x\}_{x\in R}\,) \tag{21.0.9}$$

with the degree shifts and differentials defined in a similar way. An important example include the deformed preprojective algebra in [36, 37] (see [93]).

Besides these quiver examples, there is an important class of examples of Calabi-Yau DG categories that comes from topology. Let X be any path connected pointed topological space. We say that a DG algebra A is an *Adams-Hilton model* of X, written as $A \simeq \mathsf{AH}(X)$, if A is quasi-isomorphic to the DG algebra of chains on the Moore loop space of X. Many different methods are given in the literature to provide small models for this DG algebra. See Section 25.1 for a brief review.

If X is a closed oriented manifold of dimension n, then it has a fundamental class $[X] \in \mathrm{H}_n(X)$. This class induces a Poincaré duality in its (co)homology groups. It is stated in [108] and proved in [33] that this Poincaré duality structure induces an n-Calabi-Yau structure on the Adams-Hilton model $\mathsf{AH}(X)$ of X. Thus we may view a Calabi-Yau structure as a non-commutative analogue of Poincaré duality structure.

Recall that, for a compact oriented n-dimensional manifold M with a boundary ∂M, there is also a relative Poincaré duality structure in the homology of the pair $(M, \partial M)$. Namely, there is a relative fundamental class $[M] \in \mathrm{H}_n(M, \partial M)$ that induces a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^p(M) \arrow[r] \arrow[d, "\cong"', "\cap[M]"] & \mathrm{H}^p(\partial M) \arrow[r] \arrow[d, "\cong"', "\cap[\partial M]"] & \mathrm{H}^{p+1}(M, \partial M) \arrow[r] \arrow[d, "\cong"', "\cap[M]"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_{n-p}(M, \partial M) \arrow[r] & \mathrm{H}_{n-p-1}(\partial M) \arrow[r] & \mathrm{H}_{n-p-1}(M) \arrow[r] & \cdots
\end{tikzcd} \tag{21.0.10}$$

where all the vertical maps are isomorphisms. In particular, it induces a Poincaré duality on its boundary ∂M.

It is then very natural to ask what is the non-commutative analogue of a compact oriented manifold with boundary. The answer is given by the notion of relative Calabi-Yau structures, introduced by Brav and Dyckerhoff in [15]. Let $F : \mathcal{A} \to \mathcal{B}$ be a map (*i.e.*, DG functor) between DG categories. Then one can show (see [15], as well as Section 22.3 below) that a relative Hochschild class $[\xi] \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ determines a map

$$\mathcal{A}^![n-1] \to \mathcal{A} \tag{21.0.11}$$

of $\mathcal{A}$-bimodules in the derived category $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$ of $\mathcal{A}$-bimodules, as well as a map

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{B}^![n-1] \arrow[r] \arrow[d] & \boldsymbol{L}F_!(\mathcal{A}^!)[n-1] \arrow[r] \arrow[d] & \mathrm{cone}(\mathcal{B}^! \to \boldsymbol{L}F_!(\mathcal{A}^!))[n-1] \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{cone}(\boldsymbol{L}F_!(\mathcal{A}) \to \mathcal{B})[-1] \arrow[r] & \boldsymbol{L}F_!(\mathcal{A}) \arrow[r] & \mathcal{B} \arrow[r] & \cdots
\end{tikzcd} \tag{21.0.12}$$

of distinguished triangles in the derived category $\mathcal{D}(\mathcal{B}^{\mathrm{e}})$ of $\mathcal{B}$-bimodules, where $F_! : \mathcal{C}(\mathcal{A}^{\mathrm{e}}) \to \mathcal{C}(\mathcal{B}^{\mathrm{e}})$ is the functor $F_!(M) := M \otimes_{\mathcal{A}^{\mathrm{e}}} \mathcal{B}^{\mathrm{e}}$ on bimodules induced by F.

One can view the map (21.0.12) as the non-commutative analogoue of (21.0.10). This allows one to make the following generalization of Definition 21.0.7.

Definition 21.0.13. A relative n-Calabi-Yau structure on a DG functor F : $\mathcal{A} \to \mathcal{B}$ between homologically smooth DG categories is a relative negative cyclic class $[\xi] \in \mathrm{HC}_n^-(\mathcal{B}, \mathcal{A})$ whose underlying relative Hochschild class $h([\xi]) \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ induces relative duality of bimodules. *i.e.*, the map (21.0.11) is an isomorphism in $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$, and the map (21.0.12) is an isomorphism of distinguished triangles in $\mathcal{D}(\mathcal{B}^{\mathrm{e}})$.

It was proved in [15] that if M is a compact oriented manifold with boundary ∂M, then there is a canonical relative n-Calabi-Yau structure on the map $\mathsf{AH}(\partial M) \to \mathsf{AH}(M)$ of their Adams-Hilton models. This makes precise the analogy between topology and non-commutative geometry.

The generalization of the notion of Calabi-Yau structures to the relative contexts prompts one to ask whether the quiver examples of Calabi-Yau DG categories obtained by Keller's construction of deformed Calabi-Yau completion generalizes to the relative contexts as well. Such a generalization indeed exists, and provides simple and explicit examples for DG categories with relative Calabi-Yau structures.

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a semi-free extension of semi-free DG categories. We want a construction that would extend this to a DG functor $\tilde{F} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ with a canonical relative n-Calabi-Yau structure. By definition, a relative n-Calabi-Yau structure on $\tilde{F}$ induces an (absolute) $(n-1)$-Calabi-Yau structure on $\tilde{A}$. Therefore, it is natural to take $\tilde{A}$ simply as the $(n-1)$-Calabi-Yau completion of $\mathcal{A}$. *i.e.*,

$$\tilde{A} = \Pi_{n-1}(\mathcal{A}) = T_{\mathcal{A}}(\mathrm{Res}(\mathcal{A})^{\vee}[n-2])$$

To construct $\tilde{\mathcal{B}}$, we consider the shifted cone

$$\Xi \,=\, \mathrm{cone}\left[\,\mathrm{Res}(\mathcal{B})^{\vee} \xrightarrow{\gamma_F^{\vee}} F_!(\mathrm{Res}(\mathcal{A})^{\vee})\,\right][n-2]$$

on the dual of the canonical map $\gamma_F : F_!(\mathrm{Res}(\mathcal{A})) \to \mathrm{Res}(\mathcal{B})$. We take $\tilde{\mathcal{B}}$ to be the tensor category

$$\tilde{\mathcal{B}} \,:=\, \Pi_n(\mathcal{B}, \mathcal{A}) \,:=\, T_{\mathcal{B}}(\Xi)$$

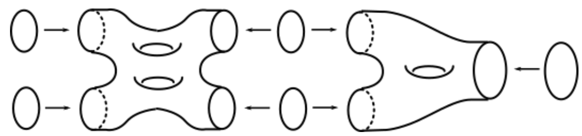
There is a canonical map $\tilde{F} : \Pi_{n-1}(\mathcal{A}) \to \Pi_n(\mathcal{B}, \mathcal{A})$ that extends F. Moreover, parallel to the absolute case, there is a natural family of deformations of this extension parametrized by relative Hochschild classes $[\eta] \in \mathrm{HH}_{n-2}(\mathcal{B}, \mathcal{A})$. The result is denoted as $\tilde{F} : \Pi_{n-1}(\mathcal{A}; \eta_{\mathcal{A}}) \to \Pi_n(\mathcal{B}, \mathcal{A}; \eta)$, and is called the *deformed relative n-Calabi-Yau completion* of $F : \mathcal{A} \to \mathcal{B}$.

Despite the apparently more involved definition, the underlying idea for the relative Calabi-Yau completion is the same as in the absolute case: one adds generating arrows in an appropriate way to ensure relative duality of bimodules on $\tilde{F}$. Thus, we have the following generalization of Theorem 21.0.8

Theorem 21.0.14. *Any negative cyclic lift of the deformation parameter $[\eta] \in \mathrm{HH}_{n-2}(\mathcal{B}, \mathcal{A})$ determines a canonical relative n-Calabi-Yau structure on the deformed relative n-Calabi-Yau completion.*

Recall that the fact that a topological space X has the structure of a compact oriented manifold is reflected at the level of noncommutative geometry by the (relative) Calabi-Yau structure on the Adams-Hilton models. One important operation one can perform on manifolds with boundary is to glue two manifolds along the same boundary. This is particularly important in corbordism theory. It is natural to ask whether this operation is also reflected in the noncommutative geometry of the Adams-Hilton models. This is indeed the case, and is worked out in [15].

Thus, imitating the situation of cobordism, we consider the following diagram



$$\mathcal{A}_3 \to \operatorname{hocolim}[\,\mathcal{Y} \leftarrow \mathcal{A}_2 \to \mathcal{X}\,] \qquad\qquad \leftarrow \mathcal{A}_1$$

The diagram suggests how to formulate a gluing procedure. The two manifolds in this diagram should be formualted as DG functors $\mathcal{A}_3 \amalg \mathcal{A}_2 \to \mathcal{Y}$ and $\mathcal{A}_2 \amalg \mathcal{A}_1 \to \mathcal{X}$, both endowed with relative n-Calabi-Yau structures. The gluing of $\mathcal{Y}$ and $\mathcal{X}$ along $\mathcal{A}_2$ should be formulated as a homotopy pushout. Then, in [15], it was shown that, if the Calabi-Yau structures are also glued along $\mathcal{A}_2$, then one obtains a relative n-Calabi-Yau structure in the resulting DG functor. $\mathcal{A}_3 \amalg \mathcal{A}_1 \to \mathcal{Y} \amalg^{\boldsymbol{L}}_{\mathcal{A}_2} \mathcal{X}$.

We will apply this algebraic gluing procedure to perform a "Calabi-Yau surgery" along a submanifold embedded inside a manifold. Thus, let N be an n-dimensional compact oriented manifold, possibly with boundary ∂N and let M be a closed oriented manifold smoothly embedded in the interior of N. For simplicity, we assume in this Introduction that M has codimension 2 in N, although the construction works when M has codimension ≥ 2. Loosely speaking, we will remove an open tubular neighborhood $\nu(M) \subset N$ of M, and glue in a DG category obtained as a relative Calabi-Yau completion. Thus, this construction combines the topological examples and the quiver examples of relative Calabi-Yau DG categories.

Let $X = N \setminus \nu(M)$ be the complement of the tubular neighborhood $\nu(M)$ in N. Then X is compact oriented, with boundary $\partial X = \partial N \cup S_M(N)$ where $S_M(N)$ is the unit normal bundle of M in N. Choose a trivialization Φ of this bundle. Then we have $\partial X = \partial N \cup M \times S^1$. Therefore, if M has r components $M = M_1 \cup \ldots \cup M_r$, then there is a canonical relative n-Calabi-Yau structure on the boundary inclusion map of the Adams-Hilton models

$$\mathsf{AH}(M_1 \times S^1) \amalg \ldots \amalg \mathsf{AH}(M_r \times S^1) \amalg \mathsf{AH}(N) \to \mathsf{AH}(X)$$

One case that is easy to visualize is when $N = \mathbb{R}^3$ and $M = L$ is a link in $\mathbb{R}^3$ with r components $L = L_1 \cup \ldots \cup L_r$. In this case, we view $\mathbb{R}^3$ as a solid ball with large enough radius, and is therefore compact. Then the link complement X will have an S^2 boundary together with r torus boundary $L_i \times S^1 \cong \mathbb{T}^2$. Along each torus boundary, we will glue in a DG category called the *perverse neighborhood* of the link component L_i. This "thickens" each torus boundary. In the general case, this means that we will glue in the perverse neighborhood $\mathscr{I}_n(M_i)$ of M_i along the boundary $M_i \times S^1$ to thicken it.

We will construct the perverse neighborhood $\mathscr{I}_n(M_i)$ by performing a relative Calabi-Yau completion. Consider the DG functor

$$F \ : \ \mathcal{A} := \mathsf{AH}(M_i) \amalg \mathsf{AH}(M_i) \to \mathsf{AH}(M_i) \otimes \vec{I} =: \mathcal{B} \tag{21.0.15}$$

where $\vec{I}$ is the k-category freely generated by the quiver $[\bullet \to \bullet]$.

Perform the relative n-Calabi-Yau completion

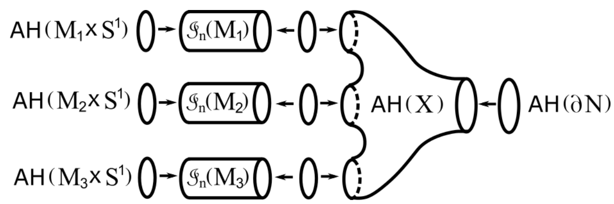
$$\tilde{F} : \Pi_{n-1}(\mathsf{AH}(M_i)) \amalg \Pi_{n-1}(\mathsf{AH}(M_i)) \to \Pi_n(\mathcal{B}, \mathcal{A})$$

of (21.0.15). It turns out that, if we localize $\Pi_{n-1}(\mathsf{AH}(M_i))$ by inverting a canonical set of degree 0 elements, the result is simply the Adams-Hilton model of $M_i \times S^1$. (See Theorem 25.2.5.) Therefore, by choosing a suitable deformation parameter $[\eta]$, we have a DG functor

$$\mathsf{AH}(M_i \times S^1) \amalg \mathsf{AH}(M_i \times S^1) \xrightarrow{(i,i')} \Pi_n^{\mathrm{loc}}(\mathcal{B}, \mathcal{A}; \eta) =: \mathscr{I}_n(M_i) \tag{21.0.16}$$

endowed with a canonical relative n-Calabi-Yau structure, where $\Pi_n^{\mathrm{loc}}(\mathcal{B}, \mathcal{A}; \eta)$ is a localization of the deformed relative n-Calabi-Yau completion of F. (See Definition 24.4.7 and 24.4.10.)

As hinted earlier, we will use the map (21.0.16) to glue the perverse neighborhood $\mathscr{I}_n(M_i)$ along the boundary of X. The gluing procedure can be described schematically by the diagram



The result of this algebraic gluing construction is then the DG category

$$\mathscr{A}(N, M; \Phi) := \operatorname{hocolim} \left[\amalg_{i=1}^r \mathscr{I}_n(M_i) \xleftarrow{i' \amalg \dots \amalg i'} \amalg_{i=1}^r \mathsf{AH}(M_i \times S^1) \xrightarrow{\mathsf{AH}(\Phi)} \mathsf{AH}(X) \right]$$

called the *perversely thickened DG category* of (N, M, Φ). As suggested by the above gluing diagram, it comes with a natural map

$$\amalg_{i=1}^r \mathsf{AH}(M_i \times S^1) \amalg \mathsf{AH}(\partial N) \to \mathscr{A}(N, M; \Phi) \tag{21.0.17}$$

that has a relative n-Calabi-Yau structure. (See Theorem 25.4.8.)

The main reason why we are interested in this construction is that, for simple examples of the pairs (N, M), the construction gives DG categories of interest in contact geometry. For example, the case when $N = D^2$ is the 2-dimensional disk and M is a set of n points in the interior of N, a calculation performed in Section 25.5 shows that the map (21.0.17) is given by

$$k\langle\mu_1^\pm\rangle \amalg \dots \amalg k\langle\mu_n^\pm\rangle \amalg k\langle Q^\pm\rangle \to \left[\begin{tikzcd} \boxed{k\langle\mu_1^\pm\rangle} \arrow[dr, bend left, "a_1"] & \cdots & \boxed{k\langle\mu_n^\pm\rangle} \arrow[dl, bend left, "a_n"] \\ & \boxed{k\langle T_1^\pm,\dots,T_n^\pm\rangle} \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_n^*"] & \end{tikzcd} \right] \Big/ \left(\begin{array}{l} \mu_i = a_i^* a \\ T_i = a_i a_i^* \end{array} \right.$$

where Q is mapped to $T_1 \cdots T_n$.

This can be viewed as the multiplicative preprojective algebra with non-central parameter. The one with central parameter can then be expressed as a homotopy pushout

$$\begin{aligned}&\mathscr{A}(D^2, \{p_1, \ldots, p_n\})|_{(t_1,\ldots,t_n,q)}\\ =&\operatorname{hocolim}\left[\, k \amalg \ldots \amalg k \amalg k \leftarrow k\langle \mu_1^{\pm}\rangle \amalg \ldots \amalg k\langle \mu_n^{\pm}\rangle \amalg k\langle Q^{\pm}\rangle \rightarrow \mathscr{A}(D^2, \{p_1, \ldots, p_n\}) \,\right]\end{aligned}$$

where the left pointing map sends the variables μ_i and Q to invertible elements $t_i \in k^\times$ and $q \in k^\times$ in the base ring k.

Notice that the 0-th homology of $\mathscr{A}(D^2, \{p_1, \ldots, p_n\})|_{(t_1,\ldots,t_n,q)}$ is precisely the multiplicative preprojective algebra [36, 37] of the star shaped quiver of all leg length 1. One can modify the gluing pattern to obtain multiplicative preprojective algebra of other graphs, as well as the higher genus versions defined in [17] (see Remark 25.5.21). The relation of this example with contact geometry was suggested in [17, 144].

For the case when $N = \mathbb{R}^3$ and $M = L$ is a link in $\mathbb{R}^3$ with a framing Φ, the DG category $\mathscr{A}(\mathbb{R}^3, L; \Phi)$ is quasi-equivalent to the link DG category constructed in [11] that extends the Legendrian DG algebra $\mathcal{A}_L$ (see [120, 121, 122, 54]) of the unit conormal bundle $ST^*_L(\mathbb{R}^3) \to ST^*(\mathbb{R}^3)$ of the link $L \subset \mathbb{R}^3$. In particular, if we denote by $\mathscr{A}(\mathbb{R}^3, L; \Phi)|_{\{1,\ldots,r\}\to\{1\}}$ the DG category obtained by collapsing the objects $\{1, \ldots, r\}$ in (25.6.21) to a single object $\{1\}$, then the endomorphism DG algebra of the DG category $\mathscr{A}(\mathbb{R}^3, L; \Phi)|_{\{1,\ldots,r\}\to\{1\}}$ at the object 1 is quasi-isomorphic to $\mathcal{A}_L$.

Both of these examples are known to be closely related to perverse sheaves. For the case of points in the 2-dimensional disk, the relation was shown in [66] (see also [17]). For the case of a link in $\mathbb{R}^3$, the relation was suggested by [56, 144] and shown in [11] (see also [10]). This relation holds in general, and can be expressed by the following theorem proved in Section 25.7 (see Theorem 25.7.11).

Theorem 21.0.18. *If k is a field, then the category $\mathtt{Mod}^{\mathrm{fd}}\big(\mathrm{H}_0(\mathscr{A}(N, M; \Phi))\big)$ of finite dimensional (left) modules of the 0-th homology of the perversely thickened DG category is equivalent to the category $\mathsf{Perv}(N, M)$ of perverse sheaves on N with singularities at most along M.*

In view of this theorem, it seems plausible to think of the category of DG modules over the perversely thickened DG category as a category of "higher perverse sheaves", in much the same way as the category of DG modules over the Adams-Hilton model $\mathsf{AH}(X)$ of a topological space X can be thought of as a category of higher local systems on X. However, this suggestion is only a speculation at the moment. It would be interesting to generalize the construction of perversely thickened DG category to other stratifications, and investigate whether this speculation can be expressed in more precise terms.

CHAPTER 22

BIMODULES, DUALITY AND HOCHSCHILD HOMOLOGY

In this section, we review some basic facts about DG categories and their bimodules. Most of the results are well-known. However, Proposition 22.3.11 and Theorem 22.5.18 appear to be new.

22.1 Bimodules and duality

Throughout Part IV, we fix a commutative ring k with unit. Unless stated otherwise, all complexes have homological grading. Unadorned tensor product will be understood to be over k. By a differential graded (DG) category over k, we mean a category $\mathcal{A}$ enriched over chain complexes $\mathcal{C}(k)$ of k. We denote the category of all (small) DG categories over k by $\texttt{dgCat}_k$. Basic notions about DG categories, DG functors, modules, *etc.* are defined, for example, in [92]. We recall some of these notions in order to set up convention.

The category $\mathcal{C}(k)$ of chain complexes over k can be enriched to a DG category $\mathcal{C}_{\mathrm{dg}}(k)$ where the Hom sets between chain complexes M and N are replaced by Hom complexes $\underline{\mathrm{Hom}}_k(M,N)$, so that $\mathcal{C}(k) = \mathrm{Z}_0(\mathcal{C}_{\mathrm{dg}}(k))$.

A right module over a DG category $\mathcal{A}$ is a DG functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$. Explicitly, a right module M associates each object $x \in \mathcal{A}$ a chain complex $M(x)$, together with product maps $M(y) \otimes \mathcal{A}(x,y) \to M(x)$ which are associative and unital in the obvious sense. Unless otherwise stated, a module will always mean a right module. We denote the category of all right modules over A as $\mathcal{C}(\mathcal{A})$, which has an obvious DG enrichment $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ where the Hom sets between DG modules M, N are replaced by Hom complexes $\underline{\mathrm{Hom}}_{\mathcal{A}}(M,N)$.

A map between modules M and N is, by definition, a homogeneous element f in the Hom-complex $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N)$. A map is said to be closed if it is closed in the Hom-complex. Thus, a degree zero closed map is simply a map in the unenriched category $\mathcal{C}(\mathcal{A})$,

For any DG categories $\mathcal{A}$ and $\mathcal{B}$, we define their tensor product $\mathcal{A} \otimes \mathcal{B}$ to be the DG category with object set $\mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ and Hom complexes

$$\underline{\mathrm{Hom}}_{\mathcal{A}\otimes\mathcal{B}}((a, b), (a', b')) := \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \otimes \underline{\mathrm{Hom}}_{\mathcal{B}}(b, b')$$

In particular, for any DG category, we define its *enveloping DG category* to be the tensor product $\mathcal{A}^{\mathrm{e}} := \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$. We define a bimodule over $\mathcal{A}$ to be a (right) module over $\mathcal{A}^{\mathrm{e}}$. This is equivalent to the following more explicit definition:

Definition 22.1.1. A *bimodule* M over a DG category $\mathcal{A}$ associates to each pair $(x, y) \in \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{A})$ of objects in $\mathcal{A}$ a chain complex $M(x, y) \in \mathcal{C}(k)$, together with maps

$$\mathcal{A}(y_1, y_2) \otimes M(x_2, y_1) \otimes \mathcal{A}(x_1, x_2) \;\to\; M(x_1, y_2) \tag{22.1.2}$$

of chain complexes, which are associative and unital in the obvious sense.

This explicit definition makes it clear that every DG category $\mathcal{A}$ is a bimodule over itself under the composition product. In particular, $\mathcal{A}^{\mathrm{e}}$ is an $\mathcal{A}^{\mathrm{e}}$–bimodule, where the bimodule action maps (22.1.2) in this case takes the form

$$[\mathcal{A}(x_3, x_4)\otimes\mathcal{A}^{\mathrm{op}}(y_2, y_1)]\otimes[\mathcal{A}(x_2, x_3)\otimes\mathcal{A}^{\mathrm{op}}(y_3, y_2)]\otimes[\mathcal{A}(x_1, x_2)\otimes\mathcal{A}^{\mathrm{op}}(y_4, y_3)] \;\to\; [\mathcal{A}(x_1, x_4)\otimes\mathcal{A}^{\mathrm{op}}(y_4, y$$

and is given by

$$(f' \otimes g') \otimes (f \otimes g) \otimes (f'' \otimes g'') \;\mapsto\; f' f f'' \otimes g'' g g'$$

where the composition in the right hand side is defined to be the ordinary composition in $\mathcal{A}$.

Thus, fixing any pair $(x, y) \in \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{A})$, the right $\mathcal{A}^{\mathrm{e}}$-module $\mathcal{A}^{\mathrm{e}}((-,-),(x,y))$ corresponds to the *inner* $\mathcal{A}$-bimodule structure of $\mathcal{A}(-,x) \otimes \mathcal{A}(y,-)$. Similarly, the left $\mathcal{A}^{\mathrm{e}}$–module $\mathcal{A}^{\mathrm{e}}((x,y),(-,-))$ corresponds to the *outer* $\mathcal{A}$-bimodule structure of $\mathcal{A}(x,-) \otimes \mathcal{A}(-,y)$.

For any $\mathcal{A}$-bimodule M, define

$$M^*(x,y) := \underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{e}}}(M(-',-''), \mathcal{A}^{\mathrm{e}}((-',-''),(x,y))) = \underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{e}}}(M(-',-''), \mathcal{A}(-',x) \otimes \mathcal{A}(y,-''))$$

When the pair (x, y) varies over the pairs of objects of $\mathcal{A}$, this defines a left $\mathcal{A}^{\mathrm{e}}$ module structure on M^*. By the above discussion, one can regard M^* as the module of $\mathcal{A}$-bilinear maps from M to $\mathcal{A}^{\mathrm{e}}$ with respect to the *inner* bimodule structure on $\mathcal{A}^{\mathrm{e}}$. Moreover, M^* inherits the *outer* bimodule structure from $\mathcal{A}^{\mathrm{e}}$.

It will be convenient to regard M^* as a right $\mathcal{A}^{\mathrm{e}}$-module instead of a left module. To this end, we consider the conjugation map $\tau : \mathcal{A}^{\mathrm{e}} \xrightarrow{\sim} (\mathcal{A}^{\mathrm{e}})^{\mathrm{op}}$, defined to be the map $(x, y) \mapsto (y, x)$ on objects, and $(f, g) \mapsto (-1)^{|f||g|}(g, f)$ on morphisms. This map is an isomorphism of DG categories, and hence induce an isomorphism $N \mapsto \overline{N}$ between the categories of left and right modules over $\mathcal{A}^{\mathrm{e}}$. In particular, we can define

Definition 22.1.3. For any bimodule $M \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{e}})$, we define its *dual bimodule* $M^\vee$ to be the right $\mathcal{A}^{\mathrm{e}}$-module (*i.e.*, $\mathcal{A}$-bimodule) given by $M^\vee := \overline{M^*}$. Explicitly, the bimodule $M^\vee$ is given by

$$M^\vee(x,y) = \underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{e}}}(M(-',-''), \mathcal{A}(-',y) \otimes \mathcal{A}(x,-''))$$

The appearances of conjugations between left and right modules over $\mathcal{A}^{\mathrm{e}}$ can be confusing when one tries to perform explicit calculations. For this reason, we give a more explicit description of the dual bimodule $M^\vee$ when the bimodule M is free.

A bimodule $M \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{e}})$ is said to be *semi-free* if there is a set of homogeneous elements $\{\xi_i \in M(x_i, y_i)\}_{i \in S}$, called a *basis* of M, such that, for any pair $(x, y) \in \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{A})$, every object $\eta \in M(x, y)$ can be written uniquely as a finite sum

$$\eta = \sum_{i \in S} f_i \cdot \xi_i \cdot g_i$$

where $g_i \in \mathcal{A}(x, x_i)$ and $f_i \in \mathcal{A}(y_i, y)$, and only finitely many of them are nonzero. When the basis set is finite, its cardinality is called the *rank* of the semi-free module M.

Let $\xi_j^\vee : M \to \mathcal{A}^{\mathrm{e}}((-', -''), (x_j, y_j))$ be the (non-closed) graded map defined by

$$\begin{array}{ccc} M(-', -'') & \xrightarrow{\xi_j^\vee} & \mathcal{A}(-', x_j) \otimes \mathcal{A}(y_j, -'') \\ \sum f_i \cdot \xi_i \cdot g_i & \mapsto & (-1)^{|f_j|(|g_j| + |\xi_j|)} g_j \otimes f_j \end{array}$$

Then $\xi_j^\vee$ is a homogeneous element in $M^\vee(y_j, x_j)$ of degree $-|\xi_j|$. The following lemma is straightforward to check.

Lemma 22.1.4. *Suppose that the bimodule M is free over a finite basis $\{\xi_1, \ldots, \xi_m\}$, then $M^\vee$ is free over the basis $\{\xi_1^\vee, \ldots, \xi_m^\vee\}$.*

This description in terms of a basis also allows one to specify the differentials of the dual bimodule $M^\vee$. Suppose we have

$$d(\xi_i) = \sum_{j=1}^{m} f_{ij} \cdot \xi_j \cdot g_{ij}$$

then the differentials on the dual basis elements are given by

$$d(\xi_j^\vee) = -\sum_{i=1}^{m} (-1)^{|f_{ij}|(|\xi_i| + |\xi_j| + |g_{ij}|)} g_{ij} \cdot \xi_i^\vee \cdot f_{ij} \tag{22.1.5}$$

Similarly, one can determine the map $\alpha^\vee : N^\vee \to M^\vee$ induced by a map $\alpha : M \to N$ in terms of basis elements. Thus, suppose M has a basis $\{\xi_1, \ldots, \xi_m\}$ and N has a basis $\{\eta_1, \ldots, \eta_n\}$, and suppose that the map α is given by

$$\alpha(\xi_i) = \sum_{j=1}^{n} f_{ij} \cdot \eta_j \cdot g_{ij}$$

then the induced map $\alpha^\vee : N^\vee \to M^\vee$ is given by

$$\alpha^\vee(\eta_j^\vee) = \sum_{i=1}^{m} (-1)^{|f_{ij}|(|\xi_i|+|\xi_j|+|g_{ij}|)} g_{ij} \cdot \xi_i^\vee \cdot f_{ij} \tag{22.1.6}$$

Definition 22.1.7. A semi-free bimodule M is said to satisfy the *Sullivan condition* if there is a basis set $\{\xi_i\}_{i \in S}$ that admits a filtration $S_1 \subset S_2 \subset \ldots$ such that $S = \bigcup S_i$ and that, for each $j \in S_i$, $i = 1, 2, \ldots$, the differential $d(\xi_j)$ lies in the submodule generated by the basis elements in S_{i-1}. Thus, in particular, $d(\xi_j) = 0$ for all $j \in S_1$.

Notice that if the DG category $\mathcal{A}$ is concentrated in non-negative degree, then every semi-free bimodule of finite rank satisfies the Sullivan condition. We will simply refer to a semi-free bimodule satisfying the Sullivan condition as a Sullivan bimodule.

Definition 22.1.8. A bimodule $M \in \mathcal{C}(\mathcal{A}^{\mathrm{e}})$ is said to be *perfect* if it is a retract in the derived category $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$ of a Suillivan bimodule of finite rank.

Remark 22.1.9. The Sullivan condition was originally defined for commutative DG algebras (see [58]). Our definition is an analogue of that condition in the case of DG bimodules. More generally, there is a notion of I-cell complexes for any model category with a class I of generating cofibrations (see, e.g. [77, 156]). The Sullivan conditions in either case is equivalent to being an I-cell complex in the respective model category. Thus, by [156, Proposition 2.2], our notion of perfect bimodules is equivalent to the one considered in [156].

Definition 22.1.10. A DG category $\mathcal{A}$ is said to be *homologically smooth* if it is perfect as a bimodule over itself.

Suppose M is a Sullivan module of finite rank, then its dual $M^\vee$ is still Sullivan of finite rank. Moreover, we have $(M^\vee)^\vee \cong M$. Therefore, if we denote the derived functor of the duality functor $(-)^\vee : \mathcal{C}(\mathcal{A}^{\mathrm{e}})^{\mathrm{op}} \to \mathcal{C}(\mathcal{A}^{\mathrm{e}})$ as

$$(-)^! : \mathcal{D}(\mathcal{A}^{\mathrm{e}})^{\mathrm{op}} \to \mathcal{D}(\mathcal{A}^{\mathrm{e}}),$$

then this functor restricts to an involutive anti-equivalence

$$(-)^! : \mathcal{D}(\mathcal{A}^{\mathrm{e}})^{\mathrm{op}}_{\mathrm{per}} \to \mathcal{D}(\mathcal{A}^{\mathrm{e}})_{\mathrm{per}},$$

in the full subcategory $\mathcal{D}(\mathcal{A}^{\mathrm{e}})_{\mathrm{per}} \subset \mathcal{D}(\mathcal{A}^{\mathrm{e}})$ of the derived category consisting of perfect objects.

22.2 Twisted complexes and convolutions

In this subsection, we introduce twisted complexes and their convolutions. These are useful tools to keep track of iterated cones and maps between them. We learned about this notion from [4], although it is probably well-known.

Let $\mathcal{A}$ be a DG category, and let $\mathcal{C}_{\mathrm{dg}} := \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ be the DG category of DG modules over $\mathcal{A}$. For any $M \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, we can define its shift $M[n] \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ by $M[n]_m := M_{m-n}$. The 'identity map' from M to $M[n]$ has degree n, and is denoted as $s^n : M \to M[n]$. Thus, elements in $M[n]$ will be denoted by $s^n x$, where $x \in M$. The differentials on $M[n]$ are then specified by the formula $d(s^n(x)) = (-1)^n s^n(d(x))$.

In general, we will always write the shift operator s explicitly when we perform shift operators, form cones, *etc*. This convention is very convenient when one has to determine the sign of each term in explicit calculations.

We follow the convention of [91] for cones. Thus, for $f : M \to N$ a closed map of DG modules $M, N \in \mathcal{C}(\mathcal{A})$, we define

$$\operatorname{cone}(f) = M[1] \oplus N, \qquad d = \begin{bmatrix} d_{M[1]} & 0 \\ fs^{-1} & d_N \end{bmatrix}$$

The distinguished triangle in the derived category $\mathcal{D}(\mathcal{A})$ associated to f is then represented by

$$\cdots \operatorname{cone}(f)[-1] \xrightarrow{\pi_0} M \xrightarrow{f} N \xrightarrow{\iota} \operatorname{cone}(f) \xrightarrow{-s\pi_0 s^{-1}} M[1] \xrightarrow{-sfs^{-1}} N[1] \to \cdots \tag{22.2.1}$$

where $\iota(y) = (0, y)$ and $\pi_0(x, s^{-1}y) = x$.

Notice that we have written the shift $f[1]$ as sfs^{-1}. More generally, given a map $f : M \to N$ of modules, we will write $s^q f s^{-p} : M[p] \to N[q]$ the corresponding shifted map.

We will denote the cone of $f : M \to N$ graphically by $\boxed{M \xrightarrow{f} N}$. Similarly, the cocone of $f : M \to N$ is defined as $\operatorname{cocone}(f) = \operatorname{cone}(f)[-1]$, and is denoted graphically by $M \xrightarrow{f} N$.

The constructions of cones and cocones associate a DG module to a closed map of DG modules. This construction can be generalized to more complicated systems of maps.

Definition 22.2.2. A *twisted complex* in $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ consists of

1. A finite collection of objects $X_0, \dots, X_n \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$.

2. A collection of homogeneous maps $f_{ij} \in \underline{\mathrm{Hom}}_{\mathcal{A}}(X_j, X_i)$ of degree $i - j - 1$.

satisfying

(a) $f_{ij} = 0 \quad$ if $j \geq i$.

(b) $d_{X_i} f_{ij} - (-1)^{i-j-1} f_{ij} d_{X_j} = \sum_{j<k<i} (-1)^{k+i-1} f_{ik} f_{kj}$.

The number n is called the *length* of the twisted complex.

Clearly, a twisted complex of length 1 is simply a closed map of DG modules. The cone construction for twisted complexes of length 1 can be generalized to the twisted complexes of arbitrary length.

Definition 22.2.3. The *convolution* of a twisted complex $\{X_*, f_{**}\}$ is the DG module whose underlying graded module is given by

$$\mathrm{Conv}(X) = X_0[n] \oplus X_1[n-1] \oplus \ldots \oplus X_n$$

with differential given by

$$d = \begin{bmatrix} d_{X_0[n]} & 0 & 0 & \cdots & 0 \\ (-1)^{n-1} s^{n-1} f_{10} s^{-n} & d_{X_1[n-1]} & 0 & \cdots & 0 \\ (-1)^{n-2} s^{n-2} f_{20} s^{-n} & (-1)^{n-2} s^{n-2} f_{21} s^{1-n} & d_{X_2[n-2]} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n0} s^{-n} & f_{n1} s^{1-n} & f_{n2} s^{2-n} & \cdots & d_{X_n} \end{bmatrix}$$

It is straightforward to check that the condition (b) in Definition 22.2.2 is equivalent to the condition that the differential d for convolution $\mathrm{Conv}(X)$ satisfy $d^2 = 0$.

Clearly, the convolution of a twisted complex of length 1 is the same as the cone of the corresponding map. For twisted complexes of higher lengths, the convolution is related to iterated cones. To see this, let $X_\bullet = (X_0, \ldots, X_n\,,\, \{f_{ij}\})$ be a twisted complex. Suppose we are given an object $X_{n+1} \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, together with maps $g_j : X_j \to X_{n+1}$ of degree $n-j$ for $j = 0, \ldots, n$. Then we can extend $X_\bullet$ to a collection $\tilde{X}_\bullet = (X_0, \ldots, X_n, X_{n+1}\,,\, \{f_{ij}\})$ with $f_{n+1,j} := g_j$. Then we have

Proposition 22.2.4. *The extension $\tilde{X}_\bullet$ is a twisted complex if and only if the map*

$$g : \mathrm{Conv}(X_\bullet) = X_0[n] \oplus X_1[n-1] \oplus \ldots \oplus X_n \xrightarrow{(g_0 s^{-n}, g_1 s^{1-n}, \ldots, g_n)} X_{n+1}$$

commutes with differentials.

Moreover, in this case, the convolution of the extension $\tilde{X}_\bullet$ is isomorphic to the cone of the map $g : \mathrm{Conv}(X_\bullet) \to X_{n+1}$.

$$\mathrm{Conv}(\tilde{X}_\bullet) = \mathrm{cone}(\mathrm{Conv}(X_\bullet) \xrightarrow{g} X_{n+1})$$

A repeated application of this proposition shows that a twisted complex $X_\bullet = (\{X_0, \ldots, X_n\}\,,\, \{f_{ij}\})$ is precisely the data required in order to perform iterated cone construction. This can be summarized graphically as

$$\mathrm{Conv}(X_\bullet) \quad = \quad \boxed{\boxed{\boxed{X_0 \longrightarrow X_1} \longrightarrow X_2} \qquad \cdots \longrightarrow X_n}$$

One can also express an iterated cocone construction as a convolution of twisted complexes. Namely, consider the shift $\mathrm{Conv}(X_\bullet)[-n]$ of the convolution, described by

$$\mathrm{Conv}(X_\bullet)[-n] = X_0 \oplus X_1[-1] \oplus \ldots \oplus X_n[-n]$$

with differentials

$$d = \begin{bmatrix} d_{X_0} & 0 & 0 & \cdots & 0 \\ -s^{-1}f_{10} & d_{X_1[-1]} & 0 & \cdots & 0 \\ (-1)^2 s^{-2} f_{20} & (-1)^2 s^{-2} f_{21} s & d_{X_2[-2]} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^n s^{-n} f_{n0} & (-1)^n s^{-n} f_{n1} s & (-1)^n s^{-n} f_{n2} s^2 & \cdots & d_{X_n[-n]} \end{bmatrix}$$

then an analogue of Proposition 22.2.4 holds. This can be summarized schematically as

$$\mathrm{Conv}(X_\bullet)[-n] \quad = \quad X_0 \longrightarrow \cdots \quad X_{n-2} \longrightarrow X_{n-1} \longrightarrow X_n$$

We will apply this mainly to the case $n = 2$. Then we have

Proposition 22.2.5. *Suppose we are given degree zero closed maps* $f : X_0 \to X_1$ *and* $g : X_1 \to X_2$ *in* $\mathcal{C}(\mathcal{A})$*. Then the following three are equivalent:*

1. *A map* $\mathrm{cone}(f) \to X_2$ *of the form*

$$\mathrm{cone}(f) = X_0[1] \oplus X_1 \xrightarrow{(hs^{-1}, g)} X_2$$

2. *A map* $X_0 \to \mathrm{cocone}(g)$ *of the form*

$$X_0 \xrightarrow{(f, -s^{-1}h)} X_1 \oplus X_2[-1] = \mathrm{cocone}(g)$$

3. *A map* $h : X_0 \to X_2$ *of degree* 1 *such that* $dh + hd = gf$*.*

22.3 Relative Hochschild homology

For bimodules $N, M \in \mathcal{C}(\mathcal{A}^{\mathrm{e}})$, one can define the chain complex $N \otimes_{\mathcal{A}^{\mathrm{e}}} \overline{M}$ where $\overline{M}$ is the left $\mathcal{A}^{\mathrm{e}}$-module conjugate to the right $\mathcal{A}^{\mathrm{e}}$-module M. For brevity of notation, we will simply write $N \otimes_{\mathcal{A}^{\mathrm{e}}} M$ for this chain complex, where M is understood to be always conjugated in such a tensor product.

Definition 22.3.1. The *Hochschild homology* of a small DG category $\mathcal{A}$ is the homology of the *Hochschild complex*

$$C_{\bullet}(\mathcal{A}) := \mathcal{A} \otimes^{\boldsymbol{L}}_{\mathcal{A}^{\mathrm{e}}} \mathcal{A}$$

Thus, if $M_{\mathcal{A}}$ is a free resolution of the bimodule $\mathcal{A}$ in $\mathcal{C}(\mathcal{A})$, and $N_{\mathcal{A}}$ is any other bimodule quasi-isomorphic to $\mathcal{A}$, then the Hochschild complex is given explicitly by $C_{\bullet}(\mathcal{A}, \mathcal{A}) = N_{\mathcal{A}} \otimes_{\mathcal{A}^{\mathrm{e}}} M_{\mathcal{A}}$.

A Hochschild class $[\xi] \in HH_n(\mathcal{A})$ determines a map $\mathcal{A}^{!}[n] \to \mathcal{A}$ in the derived category $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$. This can be seen by choosing any closed element $\xi \in N_{\mathcal{A}} \otimes_{\mathcal{A}^{\mathrm{e}}} M_{\mathcal{A}}$ of degree n in the Hochschild complex, where $M_{\mathcal{A}}$ and $N_{\mathcal{A}}$ are as in Definition (22.3.1) above. The element ξ then determines a closed map $\hat{\xi} : M_{\mathcal{A}}^{\vee} \to N_{\mathcal{A}}$ of degree n. This in turn induces a map $\hat{\xi}s^{-n} : M_{\mathcal{A}}^{\vee}[n] \to N_{\mathcal{A}}$ of degree 0, and hence represents a map $[\hat{\xi}s^{-n}] : \mathcal{A}^{!}[n] \to \mathcal{A}$ in the derived category $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$. It is straightforward to check that this map is independent of any choices.

This description can be generalized to the relative setting. Now, we investigate the effect of a DG functor on Hochschild complexes.

Let $F : \mathcal{A} \to \mathcal{B}$ be any DG functor. Denote by $F_{!} : \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{e}}) \to \mathcal{C}_{\mathrm{dg}}(\mathcal{B}^{\mathrm{e}})$ the induced functor on bimodules. *i.e.*, $F_{!}(M) := \mathcal{B} \otimes_{\mathcal{A}} M \otimes_{\mathcal{A}} \mathcal{B}$. Denote by $\boldsymbol{L}F_{!} : \mathcal{D}(\mathcal{A}^{\mathrm{e}}) \to \mathcal{D}(\mathcal{B}^{\mathrm{e}})$ the left derived functor of $F_{!}$.

Let $p_{\mathcal{A}} : M_{\mathcal{A}} \xrightarrow{\sim} \mathcal{A}$ and $p_{\mathcal{B}} : M_{\mathcal{B}} \xrightarrow{\sim} \mathcal{B}$ be bimodule resolutions of $\mathcal{A}$ and $\mathcal{B}$ respectively, then the composition $F_{!}(M_{\mathcal{A}}) \xrightarrow{F_{!}(p_{\mathcal{A}})} F_{!}\mathcal{A} \to \mathcal{B}$ lifts to a map

$\gamma_F : F_!(M_{\mathcal{A}}) \to M_{\mathcal{B}}$ of $\mathcal{B}$-bimodules. This is equivalent to giving a map $\gamma_F : M_{\mathcal{A}} \to M_{\mathcal{B}}$ of $\mathcal{A}$-bimodules.

Similarly, if we take the bimodule replacement $N_{\mathcal{A}} \xrightarrow{\sim} \mathcal{A}$ and $N_{\mathcal{B}} \xrightarrow{\sim} \mathcal{B}$ to be either $(N_{\mathcal{A}}, N_{\mathcal{B}}) = (\mathcal{A}, \mathcal{B})$ or $(N_{\mathcal{A}}, N_{\mathcal{B}}) = (M_{\mathcal{A}}, M_{\mathcal{B}})$, then there is also a map $\gamma_F : N_{\mathcal{A}} \to N_{\mathcal{B}}$ of $\mathcal{A}$-bimodule.

Together, they give a map

$$\gamma_F : C_\bullet(\mathcal{A}) = N_{\mathcal{A}} \otimes_{\mathcal{A}^e} M_{\mathcal{A}} \to N_{\mathcal{A}} \otimes_{\mathcal{B}^e} M_{\mathcal{B}} = C_\bullet(\mathcal{B}) \tag{22.3.2}$$

whose homotopy type is independent of any choices.

Recall that any $\xi_{\mathcal{A}} \in N_{\mathcal{A}} \otimes_{\mathcal{A}^e} M_{\mathcal{A}}$ induces a map $\hat{\xi_{\mathcal{A}}} : M_{\mathcal{A}}^\vee \to N_{\mathcal{A}}$ of $\mathcal{A}$-bimodules. On the other hand, the elements $\gamma_F(\xi_{\mathcal{A}}) \in N_{\mathcal{A}} \otimes_{\mathcal{B}^e} M_{\mathcal{B}}$ induces a map $\widehat{\gamma_F(\xi_{\mathcal{A}})} : M_{\mathcal{B}}^\vee \to N_{\mathcal{B}}$. These two maps are related in the following way

$$\widehat{\gamma_F(\xi_{\mathcal{A}})} = \gamma_F \circ F_!(\hat{\xi_{\mathcal{A}}}) \circ \gamma_F^\vee \, : \, M_{\mathcal{B}}^\vee \to N_{\mathcal{B}} \tag{22.3.3}$$

Definition 22.3.4. The *relative Hochschild homology* of the DG functor $f : \mathcal{A} \to \mathcal{B}$ is the homology of the *relative Hochschild complex*, defined as the cone

$$C_\bullet(\mathcal{B}, \mathcal{A}) := \operatorname{cone}(C_\bullet(\mathcal{A}) \xrightarrow{i} C_\bullet(\mathcal{B}))$$

A closed element $\xi = (s\xi_{\mathcal{A}}, \xi_{\mathcal{B}}) \in C_\bullet(\mathcal{A}, \mathcal{A})[1] \oplus C_\bullet(\mathcal{B}, \mathcal{B})) = C_\bullet(\mathcal{B}, \mathcal{A})$ of degree n consists of an element $\xi_{\mathcal{A}} \in N_{\mathcal{A}} \otimes_{\mathcal{A}^e} M_{\mathcal{A}}$ of degree $n-1$, together with an element $\xi_{\mathcal{B}} \in N_{\mathcal{B}} \otimes_{\mathcal{B}^e} M_{\mathcal{B}}$ of degree n, such that

$$d(\xi_{\mathcal{A}}) = 0 \qquad \text{and} \qquad d(\xi_{\mathcal{B}}) + \gamma_F(\xi_{\mathcal{A}}) = 0$$

The closed element $\xi_{\mathcal{A}} \in N_{\mathcal{A}} \otimes_{\mathcal{A}^e} M_{\mathcal{A}}$ determines a degree zero closed map

$$\hat{\xi_{\mathcal{A}}} s^{1-n} : M_{\mathcal{A}}^\vee[n-1] \to N_{\mathcal{A}} \tag{22.3.5}$$

The element $\xi_{\mathcal{B}} \in N_{\mathcal{B}} \otimes_{\mathcal{B}^e} M_{\mathcal{B}}$ determines a degree zero (non-closed) map

$$\hat{\xi_{\mathcal{B}}} s^{-n} : M_{\mathcal{B}}^{\vee}[n] \to N_{\mathcal{B}} \tag{22.3.6}$$

Consider the composition

$$M_{\mathcal{B}}^{\vee}[n-1] \xrightarrow{(-1)^{n-1}s^{n-1}\gamma_F^{\vee}s^{1-n}} F_!(M_{\mathcal{A}})^{\vee}[n-1] \xrightarrow{F_!(\hat{\xi_{\mathcal{A}}} s^{1-n})} F_! N_{\mathcal{A}} \xrightarrow{\gamma_F} N_{\mathcal{B}} \tag{22.3.7}$$

By (22.3.3), this composition is precisely the map $(-1)^{n-1}\widehat{\gamma_F(\xi_{\mathcal{A}})}s^{1-n} : M_{\mathcal{B}}^{\vee}[n-1] \to N_{\mathcal{B}}$ induced by the element $(-1)^{n-1}\widehat{\gamma_F(\xi_{\mathcal{A}})} \in N_{\mathcal{B}} \otimes_{\mathcal{B}^e} M_{\mathcal{B}}$.

The condition $d(\xi_{\mathcal{B}}) + \gamma_F(\xi_{\mathcal{A}}) = 0$ implies that $d((-1)^n \xi_{\mathcal{B}} s^{1-n}) = (-1)^{n-1}\gamma_F(\xi_{\mathcal{A}})s^{1-n}$. Therefore, the degree 1 map $(-1)^n \hat{\xi_{\mathcal{B}}} s^{1-n} : M_{\mathcal{B}}^{\vee}[n-1] \to N_{\mathcal{B}}$ provides a homotopy to the composition (22.3.7).

By Proposition 22.2.5, the data consisting of the maps in (22.3.7) together with the null-homotopy $(-1)^n \hat{\xi_{\mathcal{B}}} s^{1-n}$ of the composition of (22.3.7) can be viewed either as a degree zero closed map

$$\hat{\xi}' : M_{\mathcal{B}}^{\vee}[n-1] \xrightarrow{\left((-1)^{n-1}F_!(\hat{\xi_{\mathcal{A}}})\gamma_F^{\vee}s^{1-n},\,(-1)^{n-1}s^{-1}\hat{\xi_{\mathcal{B}}}s^{1-n}\right)} F_!(N_{\mathcal{A}}) \oplus N_{\mathcal{B}}[-1] = \operatorname{cone}(F_!(N_{\mathcal{A}}) \xrightarrow{i} N_{\mathcal{B}})[-1] \tag{22.3.8}$$

or as a degree zero closed map

$$\hat{\xi}'' : \operatorname{cone}(M_{\mathcal{B}}^{\vee} \xrightarrow{\gamma_F^{\vee}} F_!(M_{\mathcal{A}})^{\vee})[n-1] = M_{\mathcal{B}}^{\vee}[n] \oplus F_! M_{\mathcal{A}}^{\vee}[n-1] \xrightarrow{\left((-1)^n\hat{\xi_{\mathcal{B}}}s^{-n},\,iF_!(\hat{\xi_{\mathcal{A}}})s^{1-n}\right)} N_{\mathcal{B}} \tag{22.3.9}$$

Clearly, these can be written into a commutative diagram

$$\begin{tikzcd}
M_{\mathcal{B}}^{\vee}[n-1] \arrow[r, "(-1)^{n-1}s^{n-1}\gamma_F^{\vee}s^{1-n}"] \arrow[d, "\hat{\xi}'"] & F_!(M_{\mathcal{A}})^{\vee}[n-1] \arrow[r, "(-1)^{n-1}s^{n-1}\iota s^{1-n}"] \arrow[d, "F_!(\hat{\xi_{\mathcal{A}}})s^{1-n}"] & \operatorname{cone}(M_{\mathcal{B}}^{\vee} \xrightarrow{\gamma_F^{\vee}} F_!(\arrow[d, "(-"] \\
\operatorname{cone}(F_!(N_{\mathcal{A}}) \xrightarrow{\gamma_F} N_{\mathcal{B}})[-1] \arrow[r, "\pi_0"] & F_!(N_{\mathcal{A}}) \arrow[r, "\gamma_F"] & N_{\mathcal{B}}
\end{tikzcd} \tag{22.3.10}$$

Notice that both rows in this diagram represent maps in a distinguished triangle in the derived category $\mathcal{D}(\mathcal{B}^{\mathrm{e}})$. It is natural to ask whether (22.3.10) defines a map of distinguished triangles. The answer is affirmative, as is shown in the following

Proposition 22.3.11. *The following diagram commutes up to homotopy*

$$\begin{tikzcd}
\mathrm{cone}(M_{\mathcal{B}}^{\vee} \xrightarrow{\gamma_F^{\vee}} F_!(M_{\mathcal{A}})^{\vee})[n-1] \arrow[r, "(-1)^n s^n \pi_0 s^{-n}"] \arrow[d, "(-1)^{n-1}\hat{\xi}''"] & M_{\mathcal{B}}^{\vee}[n] \arrow[d, "s\hat{\xi}' s^{-1}"] \\
N_{\mathcal{B}} \arrow[r, "\iota"] & \mathrm{cone}(F_!(N_{\mathcal{A}}) \xrightarrow{\gamma_F} N_{\mathcal{B}})
\end{tikzcd} \tag{22.3.12}$$

where we follow the sign convention in (22.2.1).

Proof. By the definition (22.3.8) of $\hat{\xi}'$, the composition $f = (s\xi' s^{-1}) \circ ((-1)^n s^n \pi_0 s^{-n})$ is given by

$$f \ : \ \begin{tikzcd}
M_{\mathcal{B}}^{\vee}[n] \arrow[r, "-sF_!(\hat{\xi_{\mathcal{A}}})i^{\vee} s^{-n}"] \arrow[ddr, "-\hat{\xi_{\mathcal{B}}} s^{-n}"'] & F_!(N_{\mathcal{A}})[1] \\
\oplus & \oplus \\
M_{\mathcal{A}}^{\vee}[n-1] & N_{\mathcal{B}}
\end{tikzcd}$$

Similarly, by definition (22.3.9) of $\hat{\xi}''$, the composition $g = \iota \circ ((-1)^{n-1}\xi'')$ is given by

$$g \ : \ \begin{tikzcd}
M_{\mathcal{B}}^{\vee}[n] \arrow[ddr, "-\hat{\xi_{\mathcal{B}}} s^{-n}"] & F_!(N_{\mathcal{A}})[1] \\
\oplus & \oplus \\
M_{\mathcal{A}}^{\vee}[n-1] \arrow[r, "(-1)^{n-1} iF_!(\hat{\xi_{\mathcal{A}}}) s^{1-n}"'] & N_{\mathcal{B}}
\end{tikzcd}$$

Therefore, if we let $h : \mathrm{cone}(M_{\mathcal{B}}^{\vee} \xrightarrow{\gamma_F^{\vee}} F_!(M_{\mathcal{A}})^{\vee})[n-1] \to \mathrm{cone}(F_!(N_{\mathcal{A}}) \xrightarrow{\gamma_F} N_{\mathcal{B}})$ be the map of degree 1, defined by

$$h \ : \ \begin{tikzcd}
M_{\mathcal{B}}^{\vee}[n] & F_!(N_{\mathcal{A}})[1] \\
\oplus & \oplus \\
M_{\mathcal{A}}^{\vee}[n-1] \arrow[uur, "(-1)^{n-1} s\hat{\xi_{\mathcal{A}}} s^{1-n}"] & N_{\mathcal{B}}
\end{tikzcd}$$

then one has $dh + hd = g - f$. This finishes the proof. $\square$

Therefore, a relative Hochschild class $[\xi] = [(s\xi_{\mathcal{A}}, \xi_{\mathcal{B}})] \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ determines a map in $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$

$$[\hat{\xi_{\mathcal{A}}} s^{1-n}] : \mathcal{A}^![n-1] \to \mathcal{A} \tag{22.3.13}$$

as well as a map of distinguished triangles in $\mathcal{D}(\mathcal{B}^{\mathrm{e}})$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{B}^![n-1] \arrow[r] \arrow[d, "{[\hat{\xi}']}"] & \boldsymbol{L}F_!(\mathcal{A}^!)[n-1] \arrow[r] \arrow[d, "{\boldsymbol{L}F_!([\hat{\xi_{\mathcal{A}}} s^{1-n}])}"] & \mathrm{cone}(\mathcal{B}^! \xrightarrow{\gamma_F^!} \boldsymbol{L}F_!(\mathcal{A}^!))[n-1] \arrow[r] \arrow[d, "{(-1)^{n-1}[\xi'']}"] & \cdots \\
\cdots \arrow[r] & \mathrm{cone}(\boldsymbol{L}F_!(\mathcal{A}) \xrightarrow{\gamma_F} \mathcal{B})[-1] \arrow[r, "\pi_0"] & \boldsymbol{L}F_!(\mathcal{A}) \arrow[r, "\gamma_F"] & \mathcal{B} \arrow[r] & \cdots
\end{tikzcd} \tag{22.3.14}$$

Both of these maps are independent of the choices of resolutions.

22.4 Cyclic homology

We recall the notion of mixed modules, which is the necessary structure required to define cyclic homology and its variations.

Definition 22.4.1. A *mixed complex* is a graded module $M_\bullet$ over k, together with two maps $b : M_n \to M_{n-1}$ and $B : M_n \to M_{m+1}$ for each n, satisfying

$$b^2 = 0, \qquad B^2 = 0, \qquad bB + Bb = 0.$$

Given a mixed complex (M, b, B), we call the chain complex (M, b) its *underlying chain complex*, and think of the operator B as an extra structure. This extra structure is what is required to define cyclic homology.

Definition 22.4.2. The *cyclic homology* of a mixed complex $(M_\bullet, b, B)$ is defined to be the homology of the *cyclic chain complex*

$$CC(M) := \left(\frac{M((u))}{u \cdot M[[u]]} \, , \, d = b + uB \right)$$

where u is a formal variable of degree -2.

Thus, an element of degree n in the cyclic chain complex is a Laurent polynomial

$$\xi(u) \,=\, \xi^{(0)} \,+\, \xi^{(-1)}u^{-1} \,+\, \xi^{(-2)}u^{-2} \,+\, \ldots \,+\, \xi^{(-m)}u^{-m}$$

where $\xi^{(-i)}$ are elements in M of degree $|\xi^{(-i)}| = n - 2i$

There are several variations of cyclic homology. Since we will only make use of cyclic homology and negative cyclic homology in this thesis, we only recall these two notions.

Definition 22.4.3. The *negative cyclic homology* of a mixed complex $(M_\bullet, b, B)$ is defined to be the homology of the *negative cyclic chain complex*

$$CC^-(M) := (\, M[[u]]\,,\, d = b + uB\,)$$

Thus, an element of degree n in the negative cyclic chain complex is a power series

$$\xi(u) \,=\, \xi^{(0)} \,+\, \xi^{(1)}u \,+\, \xi^{(2)}u^2 \,+\, \ldots$$

is $\xi^{(i)}$ are elements in M of degree $|\xi^{(i)}| = n + 2i$

There are canonical maps $h : \mathrm{HC}^-_n(M) \to \mathrm{HH}_n(M)$, $h' : \mathrm{HH}_n(M) \to \mathrm{HC}_n(M)$ and $B : \mathrm{HC}_n(M) \to \mathrm{HC}^-_{n+1}(M)$, induced respectively by the maps

$$\begin{aligned}
&h : (\, M[[u]]\,,\, b + uB\,) \to (M, b), && \big(\xi^{(0)} + \xi^{(1)}u + \ldots\big) \mapsto \xi^{(0)} \\
&h' : (M, b) \to \left(\frac{M((u))}{u \cdot M[[u]]}\,,\, b + uB \right), && \xi \mapsto \xi \\
&B : (\, M[[u]]\,,\, b + uB\,) \to \left(\frac{M((u))}{u \cdot M[[u]]}\,,\, b + uB \right), && \big(\xi^{(0)} + \xi^{(1)}u + \ldots\big) \mapsto B(\xi^{(0)})
\end{aligned}$$

One can associate a mixed complex, known as the bar complex, to any DG category $\mathcal{A}$. We start by recalling the construction of a simplicial chain complex $\underline{C}^{\mathrm{bar}}(\mathcal{A})$, *i.e.*, a simplicial object in the category $\mathcal{C}(k)$ of chain complexes over k.

By defintiion, $\underline{C}^{\mathrm{bar}}(\mathcal{A})$ the simplicial chain complex whose m-th simplicial degree is the chain complex

$$\underline{C}^{\mathrm{bar}}_m(\mathcal{A}) := \bigoplus_{(x_0,x_1,\ldots,x_m)\in R\times\ldots\times R} \mathcal{A}(x_0,x_m)\otimes\mathcal{A}(x_1,x_0)\otimes\ldots\otimes\mathcal{A}(x_m,x_{m-1}) \quad (22.4.4)$$

with face maps induced by composition in $\mathcal{A}$, and degeneracy maps induced by identity elements in $\mathcal{A}$. See [106] for details.

The Dold-Kan correspondence establishes an equivalence between the category of simplicial objects in any abelian category with the category of nonnegatively graded chain complexes over it. Since the category $\mathcal{C}(k)$ is in particular an abelian category, one can apply the Dold-Kan normalization functor to the simplicial chain complex $\underline{C}^{\mathrm{bar}}(\mathcal{A})$. The result is a double complex $N(\underline{C}^{\mathrm{bar}}(\mathcal{A}))$. We denote the associated total complex by

$$\overline{C}^{\mathrm{bar}}(\mathcal{A}) := \mathrm{Tot}(N(\underline{C}^{\mathrm{bar}}(\mathcal{A})))$$

and call it the *reduced bar complex* of $\mathcal{A}$

Thus, the reduced bar complex is given by

$$\overline{C}^{\mathrm{bar}}(\mathcal{A}) = \bigoplus_{m\geq 0}\left(\bigoplus_{(x_0,x_1,\ldots,x_m)\in R\times\ldots\times R} \mathcal{A}(x_0,x_m)\otimes\overline{\mathcal{A}}(x_1,x_0)[1]\otimes\ldots\otimes\overline{\mathcal{A}}(x_m,x_{m-1})[1]\right) \quad (22.4.5)$$

We denote the differential in the reduced bar complex as b, then one can show (see [106]) that the reduced bar complex represents the Hochschild chain complex, *i.e.*, it is a derived tensor product as in Definition 22.3.1. In fact, one often defines the Hochschild chain complex to be the reduced bar complex.

The reduced bar complex $(\overline{C}^{\mathrm{bar}}(\mathcal{A}), b)$ can be extended to a mixed complex $(\overline{C}^{\mathrm{bar}}(\mathcal{A}), b, B)$, where the map $B : \overline{C}^{\mathrm{bar}}(\mathcal{A}) \to \overline{C}^{\mathrm{bar}}(\mathcal{A})$ is given explicitly by the formula

$$B(f_0, \overline{sf_1}, \ldots, \overline{sf_m}) = \sum_{i=0}^{m} (-1)^{(|f_0|+\ldots+|f_{i-1}|+i)(|f_i|+\ldots+|f_m|+m-i+1)} (1, \overline{sf_i}, \ldots, \overline{sf_n}, \overline{sf_0}, \ldots, \overline{sf_{i-1}}) \tag{22.4.6}$$

Definition 22.4.7. The *cyclic homology* and *negative cyclic homology* of a DG category $\mathcal{A}$ are respectively the cyclic homology and negative cyclic homology of the mixed complex $(\overline{C}^{\mathrm{bar}}(\mathcal{A}), b, B)$.

One can apply the usual constructions on chain complexes to mixed complexes. For example, given a closed map $f : M \to N$ of mixed complexes of degree zero, one can form the cone of f. This is the mixed complex whose underlying chain complex $(\,\mathrm{cone}(f), b\,)$ is the cone of the map of the underlying chain complexes $f : (M, b) \to (N, b)$. Moreover, the map B on the cone is simply the direct sum

$$B = (-sB_M s^{-1}) \oplus (B_N) \;:\; M[1] \oplus N \to M[1] \oplus N$$

Thus, the associated (negative) cyclic chain complexes of the cone is isomorphic to the cone of the associative (negative) cyclic chain complexes.

$$\mathrm{CC}(\mathrm{cone}(f)) = \mathrm{cone}[\, f : \mathrm{CC}(M) \to \mathrm{CC}(N)\,]$$

$$\mathrm{CC}^{-}(\mathrm{cone}(f)) = \mathrm{cone}[\, f : \mathrm{CC}^{-}(M) \to \mathrm{CC}^{-}(N)\,]$$

This allows one to define unambiguously the relative mixed complex for a DG functor $F : \mathcal{A} \to \mathcal{B}$. One simply defines it as the cone of the induced map $\gamma_F : \overline{C}^{\mathrm{bar}}(\mathcal{A}) \to \overline{C}^{\mathrm{bar}}(\mathcal{B})$ on the reduced bar complexes.

$$\overline{C}^{\mathrm{bar}}(\mathcal{B}, \mathcal{A}) := \mathrm{cone}\,[\,\overline{C}^{\mathrm{bar}}(\mathcal{A}) \xrightarrow{\gamma_F} \overline{C}^{\mathrm{bar}}(\mathcal{B})\,] \tag{22.4.8}$$

One can also define the tensor product of two mixed complexes. Given mixed complexes (M_1, B_1, b_1) and (M_2, B_2, b_2), we define their tensor product to be the mixed complex

$$(M_1 \otimes M_2 \, , \; b = b_1 \otimes \mathrm{id} + \mathrm{id} \otimes b_2 \, , \; B = B_1 \otimes \mathrm{id} + \mathrm{id} \otimes B_2 \,)$$

This allows one to construct a product structure in negative cyclic homology. We briefly review this construction, following [106]. First, we recall that there is a product structure on the underling chain complex of the reduced bar complex $(\overline{C}^{\mathrm{bar}}(\mathcal{A}), b)$. *i.e.*, there is a map of chain complexes

$$\mathrm{sh} \; : \; (\overline{C}^{\mathrm{bar}}(\mathcal{A}), b) \otimes (\overline{C}^{\mathrm{bar}}(\mathcal{B}), b) \; \to \; (\overline{C}^{\mathrm{bar}}(\mathcal{A} \otimes \mathcal{B}), b) \tag{22.4.9}$$

explicitly defined by a sum over shuffles (see [106]).

Since both sides have structures of mixed complexes, it is therefore natural to ask whether the map (22.4.9) is a map of mixed complexes. The answer turns out to be negative. *i.e.*, the map (22.4.9) does not commute with the maps B in the structure of mixed complexes. However, one can show that the map (22.4.9) in fact commute with B up to homotopy in the underlying chain complexes.

Namely, there is an explicitly defined map (see [106])

$$\mathrm{sh}' \; : \; \overline{C}^{\mathrm{bar}}(\mathcal{A}) \otimes \overline{C}^{\mathrm{bar}}(\mathcal{B}) \; \to \; \overline{C}^{\mathrm{bar}}(\mathcal{A} \otimes \mathcal{B}) \tag{22.4.10}$$

of degree $+2$ such that $[\mathrm{sh}, B] + [\mathrm{sh}', b] = 0$

In fact, one can furthermore show that $[\mathrm{sh}', B] = 0$. The equations

$$[b, \mathrm{sh}] = 0 \qquad\qquad [\mathrm{sh}, B] + [\mathrm{sh}', b] = 0 \qquad\qquad [\mathrm{sh}', B] = 0 \tag{22.4.11}$$

can be combined to show that the map

$$\mathsf{Sh}^- := \mathrm{sh} + u \cdot \mathrm{sh}' : \quad \big(\overline{C}^{\mathrm{bar}}(\mathcal{A}) \otimes \overline{C}^{\mathrm{bar}}(\mathcal{B}) \big)[[u]] \; \to \; \overline{C}^{\mathrm{bar}}(\mathcal{A} \otimes \mathcal{B})[[u]] \tag{22.4.12}$$

commutes with the cyclic differentials $d = b + uB$. This gives a $k[[u]]$-bilinear map on negative cyclic homology

$$\mathsf{Sh}^- : \mathrm{HC}^-(\mathcal{A}) \otimes_{k[[u]]} \mathrm{HC}^-(\mathcal{B}) \to \mathrm{HC}^-(\mathcal{A} \otimes \mathcal{B}) \tag{22.4.13}$$

Remark 22.4.14. The map (22.4.9) is an instance of monoidal Dold-Kan correspondence [145]. Given any abelian tensor category $(\mathscr{A}, \otimes)$, the category $\mathrm{Ch}^+(\mathscr{A})$ of non-negatively graded chain complexes in $\mathscr{A}$ inherits in the standard way a symmetric monoidal product, which we still denote as $\otimes$. Similarly, the category $s\mathscr{A}$ of simplicial objects in $\mathscr{A}$ inherits a symmetric monoidal product by applying the tensor product $\otimes$ levelwise. We denote this product as $\times$.

The Dold-Kan correspondence allows one to transport the symmetric monoidal product $\times$ to the category $\mathrm{Ch}^+(\mathscr{A})$ of non-negatively graded chain complexes in $\mathscr{A}$. The Eilenberg-Zilber maps [112, 145] then give maps

$$\mathrm{sh} : V \otimes W \to V \times W$$

It is easy to see that the simplicial bar complex of $\mathcal{A} \otimes \mathcal{B}$ is simply the levelwise tensor product of the simplicial bar complexes of $\mathcal{A}$ and $\mathcal{A}$.

$$\textit{i.e.,} \qquad \underline{C}^{\mathrm{bar}}(\mathcal{A} \otimes \mathcal{B}) = \underline{C}^{\mathrm{bar}}(\mathcal{A}) \times \underline{C}^{\mathrm{bar}}(\mathcal{B})$$

Applying the Eilenberg-Zilber map to the total complexes of their normalizations then give the map (22.4.9).

The cyclic shuffle map is more subtle. Its construction uses the cyclic structure of the simplicial bar complex. *i.e.,* one shows that the simplicial bar complex is naturally a cyclic object in the category of chain complexes, and use this structure to construct the cyclic shuffle map (see [106] for details).

22.5 Standard resolutions and the lifted Connes operator

For any DG category $\mathcal{A}$ with an object set R, the bimodule $\mathcal{A} \otimes_R \mathcal{A}$ defined by

$$(\mathcal{A} \otimes_R \mathcal{A})(x, y) := \bigoplus_{z \in R} \mathcal{A}(z, y) \otimes \mathcal{A}(x, z)$$

is semi-free over the set $\{1_x \otimes 1_x \in \mathcal{A}(x, x) \otimes \mathcal{A}(x, x)\}_{x \in R}$ indexed by the object set R. For notational convenience (see (22.5.7) below), we will denote by E_x the basis elements

$$E_x \;:=\; 1_x \otimes 1_x \;\in \mathcal{A}(x, x) \otimes \mathcal{A}(x, x). \tag{22.5.1}$$

The composition map in $\mathcal{A}$ defines a surjective map $m : \mathcal{A} \otimes_R \mathcal{A} \to \mathcal{A}$ of bimodules. The *bimodule of differentials* $\Omega^1(\mathcal{A})$ is defined to be the kernel of this map.

A derivation $D : \mathcal{A} \to M$ of degree p to a bimodule M is by definition a map $D : \mathcal{A}(x, y) \to M(x, y)$ of degree p for all $x, y \in R$ such that $D(fg) = D(f)g + (-1)^{p|f|} f D(g)$ for all composable morphisms f, g in $\mathcal{A}$. There is a derivation $D : \mathcal{A} \to \Omega^1(\mathcal{A})$ of degree zero defined by $D(f) := f \otimes 1_x - 1_y \otimes f$ for all $f \in \mathcal{A}(x, y)$. This derivation commutes with differential d, and is universal among all derivations of degree zero (see, e.g., [34]).

By definition of $\Omega^1(\mathcal{A})$, there is a short exact sequence of bimodules

$$0 \to \Omega^1(\mathcal{A}) \xrightarrow{\alpha} \mathcal{A} \otimes_R \mathcal{A} \xrightarrow{m} \mathcal{A} \to 0$$

where the map $\alpha : \Omega^1(\mathcal{A}) \to \mathcal{A} \otimes_R \mathcal{A}$ is given by $D(f) \mapsto f \otimes 1_x - 1_y \otimes f$ for all $f \in \mathcal{A}(x, y)$. This gives the following resolution of $\mathcal{A}$ in $\mathcal{C}(\mathcal{A}^{\mathrm{e}})$.

Definition 22.5.2. The *standard bimodule resolution* of $\mathcal{A}$ is defined to be the cone

$$\mathrm{Res}(\mathcal{A}) := \mathrm{cone}[\,\Omega^1(\mathcal{A}) \xrightarrow{\alpha} \mathcal{A} \otimes_R \mathcal{A}\,] \quad \in \mathcal{C}(\mathcal{A}^{\mathrm{e}}) \tag{22.5.3}$$

Thus, there is always a canonical quasi-isomorphism $\mathrm{Res}(\mathcal{A}) \xrightarrow{\sim} \mathcal{A}$ of bimodules. In many cases, this allows us to give a semi-free bimodule resolution of $\mathcal{A}$.

Definition 22.5.4. Let Q be a graded quiver with object set R. A DG category $\mathcal{A}$ over k is said to be *semi-free* over (R, Q) if its underlying graded k-category is freely generated by the arrows in Q over the object set R. We write this as $\mathcal{A} = T_R(Q)$.

More generally, $\mathcal{A}$ is said to be *almost semi-free* if its underlying graded k-category is a localization of a graded free category $T_R(Q)$ by a set of generating arrows $S_0 \subset Q$ of degree zero.

The following proposition is well known.

Proposition 22.5.5. *Suppose that $\mathcal{A}$ is an almost semi-free DG category, say $\mathcal{A} = T_R(Q)[S_0^{-1}]$ for a possibly empty set $S_0 \subset Q$ of generating arrows of degree zero, then the bimodule $\Omega^1(\mathcal{A})$ is semi-free over the set $\{Df\}_{f\in Q}$*

Therefore, if $\mathcal{A} = T_R(Q)[S_0^{-1}]$ is an almost semi-free, then the bimodule $\mathrm{Res}(\mathcal{A})$ is a semi-free resolution of the bimodule $\mathcal{A}$. Explicitly,

$$\mathrm{Res}(\mathcal{A}) = (\,\Omega^1(\mathcal{A})[1]\,) \oplus (\,\mathcal{A} \otimes_R \mathcal{A}\,) \tag{22.5.6}$$

is semi-free over the basis set

$$\{sDf\}_{f\in Q} \cup \{E_x\}_{x\in R} \tag{22.5.7}$$

where E_x are the basis elements in $\mathcal{A} \otimes_R \mathcal{A}$ defined in (22.5.1)

The differential d of the standard resolution $\mathrm{Res}(\mathcal{A})$ has two components $d = d_0 + d_1$, where d_0 is the differential of the (naïve) direct sum $(\,\Omega^1(\mathcal{A})[1]\,) \oplus (\,\mathcal{A} \otimes_R$

$\mathcal{A}$) of bimodules, and d_1 is the map

$$d_1 = \alpha s^{-1} : \Omega^1(\mathcal{A})[1] \to \mathcal{A} \otimes_R \mathcal{A}, \qquad d_1(f_1 \cdot sDf_2 \cdot f_3) = f_1 f_2 \cdot E_x \cdot f_3 - f_1 \cdot E_y \cdot f_2 f_3 \tag{22.5.8}$$

Now, we consider the Sullivan condition for DG categories.

Definition 22.5.9. A DG category $\mathcal{A}$ is said to be *Sullivan* (see Remark 22.1.9) if it is semi-free over some graded quiver (R, Q) that admits a filtration $Q_1 \subset Q_2 \subset \ldots$ such that every generating arrow $f \in Q_i$ has differential $d(f)$ lying in the graded category $T_R(Q_{i-1})$ generated by the subquiver Q_{i-1}.

We say that $\mathcal{A}$ is *finitely Sullivan* if the quiver (R, Q) is finite (*i.e.*, both R and Q are finite). More generally, we say that $\mathcal{A}$ is of *finite Sullivan type* if it is quasi-equivalent to a finitely Sullivan DG category.

Clearly, if $\mathcal{A}$ is Sullivan, then its standard resolution $\mathrm{Res}(\mathcal{A})$ is also Sullivan. In particular, if $\mathcal{A}$ is of finite Sullivan type, then $\mathcal{A}$ is homologically smooth.

The standard resolution can be used to give a model of the Hochschild complex. Suppose that $\mathcal{A}$ is Sullivan, then $\mathrm{Res}(\mathcal{A})$ is a Sullivan resolution of $\mathcal{A}$ as a bimodule. The Hochschild chain complex can therefore be given either as $X(\mathcal{A}) := \mathcal{A} \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{A})$, or alternatively as $\mathbb{X}(\mathcal{A}) := \mathrm{Res}(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{A})$. We call $X(\mathcal{A})$ the *X-complex* of $\mathcal{A}$, and $\mathbb{X}(\mathcal{A})$ the *double X-complex* of $\mathcal{A}$.

The X-complex $X(\mathcal{A})$ can be extended to a mixed complex. We will give a more explicit description of the X-complex and the double X-complex below. We first introduce a notation.

For any bimodule M, we denote by $M_\natural$ the tensor product $\mathcal{A} \otimes_{\mathcal{A}^e} M$. If M is semi-free over a basis set $\{\xi_i \in M(x_i, y_i)\}_{i \in S}$, then the chain complex $M_\natural$ can be described by

$$M_\natural = \bigoplus_{i \in S} \mathcal{A}(y_i, x_i) \cdot \xi_i$$

In general, $M_\natural$ can be described by

$$M_\natural = \big(\bigoplus_{z \in R} M(z,z) \big) \,/\, \big(f\xi = (-1)^{|f||\xi|} \xi f \quad \text{for } f \in \mathcal{A}(x,y),\, \xi \in M(y,x) \big)$$

Using this notation, the X-complex can be described schematically as follows

$$X(\mathcal{A}) \;=\; \big[(\,\Omega^1(\mathcal{A})[1]\,)_\natural \xrightarrow{b_1} (\,\mathcal{A} \otimes_R \mathcal{A}\,)_\natural \big] \tag{22.5.10}$$

More precisely, consider the naïve direct sum $(\,\Omega^1(\mathcal{A})[1]\,)_\natural \,\oplus\, (\,\mathcal{A} \otimes_R \mathcal{A}\,)_\natural$ of two chain complexes. Then the diagram (22.5.10) means that $X(\mathcal{A})$ is obtained by adding a differential b_1 to this naïve direct sum.

The map b_1 in (22.5.10) is defined by

$$\begin{array}{ccc} (\,\Omega^1(\mathcal{A})[1]\,)_\natural & \xrightarrow{\qquad b_1 \qquad} & (\,\mathcal{A} \otimes_R \mathcal{A}\,)_\natural \\ \| & & \| \\ \big(\bigoplus_{z \in R} \Omega^1(\mathcal{A})(z,z) \big) \,/\, \big(g\xi = (-1)^{|g||\xi|} \xi g \big) & & \bigoplus_{z \in R} \mathcal{A}(z,z) \cdot E_z \\ f_1 \cdot sDf_2 & \longmapsto & f_1 f_2 - (-1)^{|f_1||f_2|} f_2 f_1 \end{array} \tag{22.5.11}$$

then the differential b is given by $b = b_0 + b_1$.

This explicit description of the X-complex allows us to extend it to a mixed complex.

Define the *Connes operator* on the X-complex to be the map $B : X(\mathcal{A}) \to X(\mathcal{A})$ whose restriction to $(\,\Omega^1(\mathcal{A})[1]\,)_\natural$ is zero, and whose restriction to $(\,\mathcal{A} \otimes_R \mathcal{A}\,)_\natural$ is given by

$$\begin{array}{ccc} (\,\mathcal{A} \otimes_R \mathcal{A}\,)_\natural & \xrightarrow{\qquad B \qquad} & (\,\Omega^1(\mathcal{A})[1]\,)_\natural \\ \| & & \| \\ \bigoplus_{z \in R} \mathcal{A}(z,z) \cdot E_z & & \big(\bigoplus_{z \in R} \Omega^1(\mathcal{A})(z,z) \big) \,/\, \big(f\xi = (-1)^{|f||\xi|} \xi f \big) \\ f \cdot E_z & \longmapsto & sDf \end{array} \tag{22.5.12}$$

It is clear from definition that $B^2 = 0$ and $Bb_0 + b_0 B = 0$. A direct calculation also shows that $Bb_1 + b_1 B = 0$. (see, e.g., [162, 34, 138]).

This mixed complex structure on the X-complex can be used to compute (negative) cyclic homology of Sullivan DG categories.

Proposition 22.5.13. *Suppose $\mathcal{A}$ is a Sullivan DG category, then the reduced bar complex $\overline{C}^{\mathrm{bar}}(\mathcal{A})$ is quasi-isomorphic to the X-complex as a mixed complex. Therefore, the (negative) cyclic homology of $\mathcal{A}$ is isomorphic to the (negative) cyclic homology of the mixed complex $(X(\mathcal{A}), b, B)$.*

We do not know whether the double X-complex $\mathbb{X}(\mathcal{A})$ has a natural extension to a mixed complex. However, there is a closely related structure, given by a map $\tilde{B} : X(\mathcal{A}) \to \mathbb{X}(\mathcal{A})$ which lifts the Connes operator on $X(\mathcal{A})$. To see this, we first give a more explicit description of the double X-complex $\mathbb{X}(\mathcal{A})$, analogous to the description (22.5.10) of the X-complex.

This description can be summarized by the following diagram:

$$\mathbb{X}(\mathcal{A}) \quad = \quad \left[\begin{tikzcd} & \bigoplus_{x,y\in R} \mathcal{A}(x,y)\cdot E_x \cdot \mathcal{A}(y,x)\cdot E_y & \\ \bigoplus_{x\in R} \Omega^1(\mathcal{A})(x,x)[1] \arrow[ur, "b_1"] & & \bigoplus_{y\in R} \Omega^1(\mathcal{A})(y,y)[1] \arrow[ul, "b_1"'] \\ & \big(\Omega^2(\mathcal{A})[2]\big)_\natural \arrow[ul, "b_1"] \arrow[ur, "b_1"'] & \end{tikzcd} \right] \tag{22.5.14}$$

The direct sum decomposition (22.5.6) of $\mathrm{Res}(\mathcal{A})$ induces a direct sum decomposition of the double X-complex $\mathbb{X}(\mathcal{A}) = \mathrm{Res}(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{A})$ into four components, each with a differential b_0. They are identified with each of the entries in the diagram (22.5.14). More explicitly, we have

$$\begin{aligned}
(\,\mathcal{A}\otimes_R\mathcal{A}\,)\otimes_{\mathcal{A}^{\mathrm{e}}}(\,\mathcal{A}\otimes_R\mathcal{A}\,) &= \bigoplus_{x,y\in R}\mathcal{A}(x,y)\cdot E_x\cdot\mathcal{A}(y,x)\cdot E_y\\
(\,\mathcal{A}\otimes_R\mathcal{A}\,)\otimes_{\mathcal{A}^{\mathrm{e}}}(\,\Omega^1(\mathcal{A})[1]\,) &= \bigoplus_{x\in R}\Omega^1(\mathcal{A})(x,x)[1]\\
(\,\Omega^1(\mathcal{A})[1]\,)\otimes_{\mathcal{A}^{\mathrm{e}}}(\,\mathcal{A}\otimes_R\mathcal{A}\,) &= \bigoplus_{y\in R}\Omega^1(\mathcal{A})(y,y)[1]\\
(\,\Omega^1(\mathcal{A})[1]\,)\otimes_{\mathcal{A}^{\mathrm{e}}}(\,\Omega^1(\mathcal{A})[1]\,) &= \big(\,\Omega^2(\mathcal{A})[2]\,\big)_\natural
\end{aligned}\tag{22.5.15}$$

Moreover, the extra differential d_1 of $\mathrm{Res}(\mathcal{A})$ (see (22.5.8)) induces extra differentials b_1 between different components. This is summarized in (22.5.14).

Define the map $\tilde{B}\,:\,X(\mathcal{A})\,\to\,\mathbb{X}(\mathcal{A})$ on each component of the X-complex. On the component $(\,\mathcal{A}\otimes_R\mathcal{A}\,)_\natural$, the map $\tilde{B}$ is defined by

$$\begin{array}{ccc}
\bigoplus_{z\in R}\mathcal{A}(z,z)\cdot E_z & \longrightarrow & \big(\bigoplus_{x\in R}\Omega^1(\mathcal{A})(x,x)[1]\,\big)\,\oplus\,\Big(\bigoplus_{y\in R}\Omega^1(\mathcal{A})(y,y)[1]\,\Big)\\
f & \longmapsto & (\qquad sDf\qquad,\qquad sDf\qquad)
\end{array}\tag{22.5.16}$$

On the component $(\,\Omega^1(\mathcal{A})[1]\,)_\natural$, the map $\tilde{B}$ is defined by

$$\begin{array}{ccc}
\bigoplus_{z\in R}\big(\,\Omega^1(\mathcal{A})[1]\,\big)/([f,\xi]) & \longrightarrow & \bigoplus_{z\in R}\big(\,\Omega^2(\mathcal{A})[2]\,\big)/([f,\xi])\\
f_1\cdot sDf_2\cdot f_3 & \longmapsto & \begin{array}{c}(-1)^{|f_3|(|f_1|+|f_2|+1)}\,sD(f_3f_1)\cdot sDf_2\\ +\,(-1)^{|f_1|(|f_2|+|f_3|+1)+|f_2|+1}\,sDf_2\cdot sD(f_3f_1)\end{array}
\end{array}\tag{22.5.17}$$

Then we have

Theorem 22.5.18. *The formulas* (22.5.16) *and* (22.5.17) *give a well-defined map* $\tilde{B}\,:\,X(\mathcal{A})\,\to\,\mathbb{X}(\mathcal{A})$ *of degree* 1*. This map anti-commute with the b-differentials. i.e., we have* $b\tilde{B}+\tilde{B}b=0$*. Moreover, let* $p_{\mathcal{A}}\,:\,\mathbb{X}(\mathcal{A})\,\to\,X(\mathcal{A})$ *be the canonical projection map, then we have*

$$p_{\mathcal{A}}\circ\tilde{B}=B\;:\;X(\mathcal{A})\,\to\,X(\mathcal{A})$$

Proof. It is clear that (22.5.16) is well-defined. To show that (22.5.17) is well-defined, we need to check two equations

$$\tilde{B}(f_1 \cdot sDf_2 \cdot f_3 f_4) = (-1)^{|f_4|(|f_1|+|f_2|+|f_3|+1)}\, \tilde{B}(f_4 f_1 \cdot sDf_2 \cdot f_3) \tag{22.5.19}$$

$$\tilde{B}(f_1 \cdot sD(f_2 f_3) \cdot f_4) = \tilde{B}(f_1 \cdot sDf_2 \cdot f_3 f_4) + (-1)^{|f_2|}\tilde{B}(f_1 f_2 \cdot sDf_3 \cdot f_4) \tag{22.5.20}$$

for all composible morphisms f_1, f_2, f_3, f_4. It is easy to see that the equation (22.5.19) holds. Therefore, to show (22.5.20), one suffices to assume $f_4 = 1$. A direct computation will then show that the two sides are equal in $\big(\,\Omega^2(\mathcal{A})[2]\,\big)_\natural$.

Recall that the b-differentials of each of $X(\mathcal{A})$ and $\mathbb{X}(\mathcal{A})$ is a sum of two parts $b = b_0 + b_1$. It is clear from the definition of $\tilde{B}$ that we have $b_0\tilde{B} + \tilde{B}b_0 = 0$. Moreover, a direct calculation will show that $b_1\tilde{B} + \tilde{B}b_1 = 0$.

The last statement is obvious. □

Theorem 22.5.18 shows that the map $\tilde{B} : X(\mathcal{A}) \to \mathbb{X}(\mathcal{A})$ is a lift of the Connes operator $B : X(\mathcal{A}) \to X(\mathcal{A})$. For this reason, we call $\tilde{B}$ the *lifted Connes operator*.

CHAPTER 23

RELATIVE CALABI-YAU COMPLETION

In this section, we give a universal construction, called relative Calabi-Yau completion, that extends any given DG functor $F : \mathcal{A} \to \mathcal{B}$ to a DG functor $\tilde{F} : \Pi_{n-1}(\mathcal{A}) \to \Pi_n(\mathcal{B}, \mathcal{A})$, together with a family of deformations of $\tilde{F}$ parametrized by relative negative cyclic homology classes $[\eta] \in \mathrm{HC}^-_{n-2}(\mathcal{B}, \mathcal{A})$. We show that, under a finiteness condition, these extensions have canonical relative Calabi-Yau structures in the sense of [15].

23.1 Relative Calabi-Yau structure

First, we recall the notion of an (absolute) Calabi-Yau structure.

Definition 23.1.1. A *weak (absolute) n-Calabi-Yau structure* on a homologically smooth DG category $\mathcal{A}$ is a Hochschild class $[\xi_{\mathcal{A}}] \in \mathrm{HH}_n(\mathcal{A})$ such that the induced map $[\hat{\xi_{\mathcal{A}}} s^{-n}] : \mathcal{A}^![n] \to \mathcal{A}$ is an isomorphism in $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$.

A DG category having a weak n-Calabi-Yau structure was simply said to be n-Calabi-Yau in [68]. However, it was long thought that the data contained in a weak n-Calabi-Yau structure should be enriched One such enrichment was suggested in [94], and adopted in [15].

Definition 23.1.2. An *(absolute) n-Calabi-Yau structure* on a homologically smooth DG category $\mathcal{A}$ is a negative cyclic class $[\tilde{\xi_{\mathcal{A}}}] \in \mathrm{HC}^-_n(\mathcal{A})$ whose underlying Hochschild class $[\xi_{\mathcal{A}}] := h([\tilde{\xi_{\mathcal{A}}}]) \in \mathrm{HH}_n(\mathcal{A})$ is a weak n-Calabi-Yau structure.

Following [15], this notion has a generalization to the relative contexts.

Given a DG functor $F : \mathcal{A} \to \mathcal{B}$ between homologically smooth DG categories, recall that a relative Hochschild class $[\xi] \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ induces a map (22.3.13) in the derived category $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$, as well as a map (22.3.14) of distinguished triangles in $\mathcal{D}(\mathcal{B}^{\mathrm{e}})$.

Definition 23.1.3. A *weak relative n-Calabi-Yau structure* on a DG functor $F : \mathcal{A} \to \mathcal{B}$ between homologically smooth DG categories is a relative Hochschild class $[\xi] \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ such that

1. The induced map (22.3.13) is an isomorphism in $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$; and
2. the induced map (22.3.14) is an isomorphism of distinguished triangles in $\mathcal{D}(\mathcal{B}^{\mathrm{e}})$.

As in the absolute case, we consider enrichments of weak Calabi-Yau structures to relative negative cyclic classes.

Definition 23.1.4. An n-Calabi-Yau structure on F is a relative negative cyclic class $[\tilde{\xi}] \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ whose underlying relative Hochschild class $h([\tilde{\xi}]) \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ is a weak relative n-Calabi-Yau structure.

Remark 23.1.5. In [15], two notions of (relative) Calabi-Yau structures are defined, called *left* (relative) Calabi-Yau structures, and *right* (relative) Calabi-Yau structures. We will always work with left (relative) Calabi-Yau structures in this thesis. Therefore, we have chosen to omit the word 'left' throughout our discussion.

Definition 23.1.4 generalizes Definitoin 23.1.1 and 23.1.2. Precisely, let ϕ be the empty DG category, then ϕ is the initial object in the category of all (small) DG category. Absolute n-Calabi-Yau structures on a homologically smooth DG category $\mathcal{B}$ is in canonical bijection with relative n-Calabi-Yau structures on the (unique) map $\phi \to \mathcal{B}$.

23.2 Relative Calabi-Yau completion

Given a DG category $\mathcal{A}$ and a bimodule $M \in \mathcal{C}(\mathcal{A}^{\mathrm{e}})$, we denote by $T_{\mathcal{A}}(M)$ the tensor category

$$T_{\mathcal{A}}(M) \;=\; \mathcal{A} \,\oplus\, M \,\oplus\, M \otimes_{\mathcal{A}} M \,\oplus\, \dots \tag{23.2.1}$$

The formula (23.2.1) defines, a priori, a bimodule on $\mathcal{A}$. We furthermore declare that $T_{\mathcal{A}}(M)$ has a DG category structure, given by concatenation product, such that the bimodule (23.2.1) is the bimodule associated to the canonical inclusion map $\mathcal{A} \to T_{\mathcal{A}}(M)$ of DG categories.

Suppose instead that M' is an element in the derived category $M' \in \mathcal{D}(\mathcal{A}^{\mathrm{e}})$, then by the tensor category $T_{\mathcal{A}}(M')$, we mean the tensor category $T_{\mathcal{A}}(M)$ of any cofibrant representative $M \in \mathcal{C}(\mathcal{A}^{\mathrm{e}})$ of M'. It is easy to show that, up to quasi-equivalence in the under category $\mathcal{A} \downarrow \mathtt{dgCat}_k$, the tensor category $T_{\mathcal{A}}(M)$ is independent of the choice of the cofibrant representative M.

Moreover, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be k-flat DG categories. *i.e.*, all the Hom complexes in $\mathcal{A}_1$ and $\mathcal{A}_2$ are flat as chain complexes over k. (This is automatic if k is a field.) Suppose that $F : \mathcal{A}_1 \to \mathcal{A}_2$ is a quasi-equivalence, then for any cofibrant bimodule $M_1 \in \mathcal{C}(\mathcal{A}_1^{\mathrm{e}})$, the canonical map $\tilde{F} : T_{\mathcal{A}_1}(M_1) \to T_{\mathcal{A}_2}(F_!(M_2))$ is a quasi-equivalence. (This follows, e.g., from Propositions A.0.4 and A.0.8).

We recall the notion of an (absolute) Calabi-Yau completion, defined in [93].

Definition 23.2.2. The (absolute) *m-Calabi-Yau completion* of a k-flat DG category $\mathcal{A}$ is defined as

$$\Pi_m(\mathcal{A}) := T_{\mathcal{A}}(\mathcal{A}^![m-1])$$

Keller claimed in [93] that the m-Calabi-Yau completion of any homologically smooth and k-flat DG category has a (weak) m-Calabi-Yau structure. While we think this is true, we do not understand the proof of this claim in [93]. Instead, we will prove the following theorem under a finiteness condition.

Theorem 23.2.3. *Suppose that $\mathcal{A}$ is of finite Sullivan type, then its m-Calabi-Yau completion $\Pi_m(\mathcal{A})$ has a canonical m-Calabi-Yau completion.*

We extend this construction to the relative setting. Thus, let $F : \mathcal{A} \to \mathcal{B}$ be a DG functor between homologically smooth and k-flat DG categories. Consider the following $\mathcal{B}$-bimodule

$$\Xi := \mathrm{cone}(\, \mathcal{B}^! \xrightarrow{i^!} \boldsymbol{L}F_!(\mathcal{A}^!)\,)[n-2] \tag{23.2.4}$$

in the derived category $\mathcal{D}(\mathcal{B}^{\mathrm{e}})$.

The canonical map $\boldsymbol{L}F_!(\mathcal{A}^!)[n-2] \to \Xi$ in $\mathcal{D}(\mathcal{B}^{\mathrm{e}})$ can be viewed as a map $\mathcal{A}^![n-2] \to \Xi$ where Ξ is now viewed as an $\mathcal{A}$-bimodule in the derived category $\mathcal{D}(\mathcal{A}^{\mathrm{e}})$. This determines a map

$$T_{\mathcal{A}}(\mathcal{A}^![n-2]) \to T_{\mathcal{B}}(\Xi) \tag{23.2.5}$$

Notice that $T_{\mathcal{A}}(\mathcal{A}^![n-2])$ is simply the absolute $(n-1)$-Calabi-Yau completion of $\mathcal{A}$. Denote by $\Pi_n(\mathcal{B}, \mathcal{A})$ the tensor category $T_{\mathcal{B}}(\Xi)$, then the map (23.2.5) can be rewritten as

$$\tilde{F} : \Pi_{n-1}(\mathcal{A}) \to \Pi_n(\mathcal{B}, \mathcal{A}) \tag{23.2.6}$$

Definition 23.2.7. The DG functor (23.2.6) is called the *relative n-Calabi-Yau completion* of F.

Theorem 23.2.3 can be generalized to the relative case.

Theorem 23.2.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be a DG functor between k-flat DG categories of finite Sullivan type, then the relative n-Calabi-Yau completion $\tilde{F} : \Pi_{n-1}(\mathcal{A}) \to \Pi_n(\mathcal{B}, \mathcal{A})$ has a canonical relative n-Calabi-Yau structure.*

In [93], Keller constructed a natural family of deformations of the (absolute) Calabi-Yau completion $\Pi_m(\mathcal{A})$, parametrized by the elements in the Hochschild homology group $\eta_{\mathcal{A}} \in \mathrm{HH}_{m-2}(\mathcal{A})$.

Let $\mathcal{A}$ be homologically smooth and k-flat, then $\mathcal{A}$ has a resolution by a finitely Sullivan bimodule $M_{\mathcal{A}} \in \mathcal{C}(\mathcal{A}^{\mathrm{e}})$. By definition, the absolute m-Calabi-Yau completion is given by $\Pi_m(\mathcal{A}) = T_{\mathcal{A}}(M_{\mathcal{A}}^{\vee}[m-1])$. As we discussed in Section 22.3, any Hochschild class $[\eta_{\mathcal{A}}] \in \mathrm{HH}_{m-2}(\mathcal{A})$ determines a closed map of bimodules

$$\hat{\eta_{\mathcal{A}}} s^{1-m} \, : \, M_{\mathcal{A}}^{\vee}[m-1] \, \to \, \mathcal{A} \tag{23.2.9}$$

of degree -1.

Let d_0 denote the differential in the DG category $T_{\mathcal{A}}(M_{\mathcal{A}}^{\vee}[m-1])$, and let d_1 denote the unique derivation on $T_{\mathcal{A}}(M_{\mathcal{A}}^{\vee}[m-1])$ that is zero on $\mathcal{A}$ and is given by the map (23.2.9) on $M_{\mathcal{A}}^{\vee}[m-1]$. It is clear that $d_1^2 = 0$. Since the map (23.2.9) is closed, we also have $d_1 d_0 + d_0 d_1 = 0$. Therefore, the derivation $d = d_{\eta_{\mathcal{A}}} := d_0 + d_1$ on $T_{\mathcal{A}}(M_{\mathcal{A}}^{\vee}[m-1])$ satisfies $d^2 = 0$. This defines a DG category

$$\Pi_m(\mathcal{A}\,;\,\eta_{\mathcal{A}}) = \big(T_{\mathcal{A}}(M_{\mathcal{A}}^{\vee}[m-1])\,,\,d\big)$$

Definition 23.2.10. The DG category $\Pi_m(\mathcal{A}\,;\,\eta_{\mathcal{A}})$ is called the *deformed m-Calabi-Yau completion* of $\mathcal{A}$ with respect to the Hochschild class $[\eta_{\mathcal{A}}] \in \mathrm{HH}_{m-2}(\mathcal{A})$.

The deformed m-Calabi-Yau completion depends only on the Hochschild homology class $[\eta_{\mathcal{A}}]$, and is independent of the choice of the closed element in $\mathcal{A} \otimes_{\mathcal{A}^{\mathrm{e}}} M_{\mathcal{A}}$ representing it. To see this, suppose $\eta_{\mathcal{A}} + d(\zeta_{\mathcal{A}})$ is any other closed class representing the Hochschild class, then consider the map

$$\Phi : \big(\, T_{\mathcal{A}}(M_{\mathcal{A}}^{\vee}[m-1])\,,\, d_{\eta_{\mathcal{A}}} \,\big) \;\to\; \big(\, T_{\mathcal{A}}(M_{\mathcal{A}}^{\vee}[m-1])\,,\, d_{\eta_{\mathcal{A}}+d(\zeta_{\mathcal{A}})} \,\big) \qquad (23.2.11)$$

such that $\Phi|_{\mathcal{A}}$ is the identity, and $\Phi|_{M_{\mathcal{A}}^{\vee}[m-1]}$ is given by

$$(\, \mathrm{id}\,,\, \zeta_{\mathcal{A}} s^{1-m} \,) \,:\, M_{\mathcal{A}}^{\vee}[m-1] \;\to\; M_{\mathcal{A}}^{\vee}[m-1] \,\oplus\, \mathcal{A}$$

Then it is straightforward to show that this map commutes with differentials and gives an isomorphism of DG categories.

Keller claimed in [93, Theorem 5.2] that the deformed m-Calabi-Yau completion always have a (weak) m-Calabi-Yau structure. We give a counter-example to this claim at the end of Section 23.3. Instead, we show that, if the deformation parameter $[\eta_{\mathcal{A}}] \in \mathrm{HH}_{m-2}(\mathcal{A})$ has a negative cyclic lift $[\tilde{\eta_{\mathcal{A}}}] \in \mathrm{HC}^{-}_{m-2}(\mathcal{A})$, then the deformed m-Calabi-Yau completion does indeed have a canonical m-Calabi-Yau structure. This is an illustration of the importance of negative cyclic enrichments.

Theorem 23.2.12. *Any negative cyclic lift $[\tilde{\eta_{\mathcal{A}}}] \in \mathrm{HC}^{-}_{m-2}(\mathcal{A})$ of the deformation parameter $[\eta_{\mathcal{A}}]$ determines an m-Calabi-Yau structure on the deformed m-Calabi-Yau completion $\Pi_m(\mathcal{A};\eta_{\mathcal{A}})$.*

This construction can be extended to the relative case. Suppose $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ are Sullivan bimodule replacement of $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{C}(\mathcal{A}^{\mathrm{e}})$ and $\mathcal{C}(\mathcal{B}^{\mathrm{e}})$ respectively.

Let $\eta = (s\eta_{\mathcal{A}}, \eta_{\mathcal{B}}) \in \operatorname{cone}(\mathcal{A} \otimes M_{\mathcal{A}} \to \mathcal{B} \otimes M_{\mathcal{B}})$ be a closed element of degree $n-2$.

In Section 22.3, we have seen that this determines a closed map

$$\hat{\eta_{\mathcal{A}}} s^{2-n} : M_{\mathcal{A}}^{\vee}[n-2] \to \mathcal{A} \tag{23.2.13}$$

of degree -1, as well as a commutative diagram of closed maps

$$\begin{tikzcd} F_!(M_{\mathcal{A}})^{\vee}[n-2] \arrow[r, "(-1)^{n-2}s^{n-2}\iota s^{2-n}"] \arrow[d, "F_!(\hat{\eta_{\mathcal{A}}})s^{2-n}"] & \operatorname{cone}\big[\, M_{\mathcal{B}}^{\vee} \xrightarrow{\gamma_F^{\vee}} F_!(M_{\mathcal{A}})^{\vee} \,\big][n-2] \arrow[d, "(-1)^{n-2}\hat{\eta}''"] \\ F_!(\mathcal{A}) \arrow[r, "\gamma_F"] & \mathcal{B} \end{tikzcd} \tag{23.2.14}$$

where the vertical maps have degree -1

Thus, we can use the map (23.2.13) to deform the differential in the (absolute) $(n-1)$-Calabi-Yau complement of $\mathcal{A}$. Moreover, since $\Xi = \operatorname{cone}(M_{\mathcal{B}}^{\vee} \xrightarrow{\gamma_F^{\vee}} F_!(M_{\mathcal{A}})^{\vee})[n-2]$, we can also use the vertical map $\hat{\eta}''$ in (23.2.14) to deform the differential in $T_{\mathcal{B}}(\theta)$.

Precisely, let d_0 be the differential of $T_{\mathcal{B}}(\theta)$, and let d_1 be the derivation of degree -1 that is zero on $\mathcal{B}$ and is given by the map $(-1)^{n-2}\hat{\eta}'' : \Xi \to \mathcal{B}$ on Ξ, then the derivation $d = d_\eta := d_0 + d_1$ satisfies $d^2 = 0$. We denote by the DG category endowed with the deformed differential d_η by

$$\Pi_n(\mathcal{B}, \mathcal{A}; \eta) := \big(\, T_{\mathcal{B}}(\Xi)\,,\, d_\eta \,\big) \tag{23.2.15}$$

Then, by the commutativity of the diagram (23.2.14), the canonical map

$$\tilde{F} : \Pi_{n-1}(\mathcal{A}; \eta_{\mathcal{A}}) \to \Pi_n(\mathcal{B}, \mathcal{A}; \eta) \tag{23.2.16}$$

commutes with the differentials, and hence is a DG functor.

Definition 23.2.17. The DG functor (23.2.16) is called the *deformed relative n-Calabi-Yau completion* of $F : \mathcal{A} \to \mathcal{B}$ with respect to the relative Hochschild class $[\eta] \in \mathrm{HH}_{n-2}(\mathcal{B}, \mathcal{A})$.

As in the absolute case, it can be shown that the deformed relative n-Calabi-Yau completion depends only on the relative Hochschild class $[\eta]$, and not on the particular choice of closed element representing it.

Theorem 23.2.12 can be generalized to the relative case.

Theorem 23.2.18. *Let $F : \mathcal{A} \to \mathcal{B}$ be a DG functor between k-flat DG categories of finite Sullivan type, and let $[\tilde{\eta}] \in \mathrm{HC}^{-}_{n-2}(\mathcal{B}, \mathcal{A})$ be a relative negative cyclic class. Denote by $[\eta]$ the image of $[\tilde{\eta}]$ under $h : \mathrm{HC}^{-}_{n-2}(\mathcal{B}, \mathcal{A}) \to \mathrm{HH}_{n-2}(\mathcal{B}, \mathcal{A})$. Then the deformed relative n-Calabi-Yau completion $\tilde{F} : \Pi_{n-1}(\mathcal{A}; \eta_{\mathcal{A}}) \to \Pi_n(\mathcal{B}, \mathcal{A}; \eta)$ has a canonical relative n-Calabi-Yau structure.*

In view of Theorem 23.2.12 and 23.2.18, we think of the (relative) negative cyclic classes as the "true" deformation parameters.

Theorem 23.2.3 and 23.2.12 will be proved in Section 23.3, while Theorem 23.2.8 and 23.2.18 will be proved in Section 23.4.

23.3 The absolute case

Consider a finitely Sullivan DG category $\mathcal{A} = T_R(Q)$, where (R, Q) is a finite graded quiver. We first give a quiver description of its (absolute) m-Calabi-Yau completion.

By Proposition 22.5.5, the canonical bimodule resolution $\mathrm{Res}(\mathcal{A})$ of $\mathcal{A}$ is a finitely Sullivan resolution, with bases (22.5.7). Therefore, the bimodule $\mathrm{Res}(\mathcal{A})^\vee[m-1]$ is Sullivan of finite rank, with basis

$$\{s^{m-1}(E_{x,\mathcal{A}})^\vee\}_{x\in R} \cup \{s^{m-1}(sDf_{\mathcal{A}})^\vee\}_{f\in Q} \tag{23.3.1}$$

where we have added subscripts $\mathcal{A}$ to avoid confusion in the relative case where the analogous basis elements also arise for $\mathcal{B}$.

Rename these basis elements as

$$f_{\mathcal{A}}^{\vee} := s^{m-1}(sDf_{\mathcal{A}})^{\vee} \qquad \text{and} \qquad c_{x,\mathcal{A}} := s^{m-1}(E_{x,\mathcal{A}})^{\vee} \tag{23.3.2}$$

Thus, the element $f_{\mathcal{A}}^{\vee}$ have degree $|f_{\mathcal{A}}^{\vee}| = m - 2 - |f|$, and points in the opposite direction to f. *i.e.*, if $f : x \to y$, then we have $f_{\mathcal{A}}^{\vee} : y \to x$. The element $c_{x,\mathcal{A}}$ has degree $|c_{x,\mathcal{A}}| = m - 1$. and points from x to x.

Then we have

$$\Pi_m(\mathcal{A}) = T_{\mathcal{A}}(\mathrm{Res}(\mathcal{A})^{\vee}[m-1]) = T_R\big(\{f\}_{f\in Q} \cup \{f_{\mathcal{A}}^{\vee}\}_{f\in Q} \cup \{c_{x,\mathcal{A}}\}_{x\in R}\big) \tag{23.3.3}$$

Now, we construct an (absolute) m-Calabi-Yau structure on the Calabi-Yau completion $\Pi_m(\mathcal{A})$. There are two ingredients in this construction. The first is the Casimir element.

Consider the element $\theta_{\mathcal{A}} \in \mathrm{Res}(\mathcal{A})^{\vee} \otimes_{\mathcal{A}^{\mathrm{e}}} \mathrm{Res}(\mathcal{A})$ corresponding to the identity map $\mathrm{id} : \mathrm{Res}(\mathcal{A})^{\vee} \to \mathrm{Res}(\mathcal{A})^{\vee}$. Thus, $\theta_{\mathcal{A}}$ is closed of degree zero, and is given explicitly as

$$\theta_{\mathcal{A}} = \sum_{f\in Q}(-1)^{(|f|+1)^2}(sDf_{\mathcal{A}})^{\vee} \otimes sDf + \sum_{x\in R}(E_{x,\mathcal{A}})^{\vee} \otimes E_x \tag{23.3.4}$$

We call this the *Casimir element* of $\mathcal{A}$.

The second ingredient in the construction an m-Calabi-Yau structure on the Calabi-Yau completion $\Pi_m(\mathcal{A})$ is a map

$$j : \mathrm{Res}(\mathcal{A})^{\vee}[m-1] \otimes_{\mathcal{A}^{\mathrm{e}}} \mathrm{Res}(\mathcal{A}) \to \Pi_m(\mathcal{A}) \otimes_{\Pi_m(\mathcal{A})^{\mathrm{e}}} \mathrm{Res}(\Pi_m(\mathcal{A})) = X(\Pi_m(\mathcal{A})) \tag{23.3.5}$$

The map $i_{\mathcal{A}} : \mathcal{A} \to \Pi_m(\mathcal{A})$ induces a map $\gamma_{\mathcal{A}} : \mathrm{Res}(\mathcal{A}) \to \mathrm{Res}(\Pi_m(\mathcal{A}))$ of $\mathcal{A}$-bimodules. Moreover, the tensor category $\Pi_m(\mathcal{A}) = T_{\mathcal{A}}(\mathrm{Res}(\mathcal{A})^\vee[m-1])$ contains, by definition, the $\mathcal{A}$-bimodules $\Pi_m(\mathcal{A})$. These two $\mathcal{A}$-bilinear maps induce the map (23.3.5) on their tensor product.

These two ingredients can be combined to construct an m-Calabi-Yau structure on $\Pi_m(\mathcal{A})$. First, we shift the Casimir element

$$s^{m-1}\theta_{\mathcal{A}} = \sum_{f\in Q}(-1)^{(|f|+1)^2} s^{m-1}(sDf_{\mathcal{A}})^\vee \otimes sDf + \sum_{x\in R} s^{m-1}(E_{x,\mathcal{A}})^\vee \otimes E_x$$

which is then an element in the chain complex $\mathrm{Res}(\mathcal{A})^\vee[m-1] \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{A})$, and is closed of degree $m-1$.

In the undeformed case, the map (23.3.5) is a chain map. Therefore, the image $j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}}) \in X(\Pi_m(\mathcal{A}))$ is a closed element of degree $m-1$ in the X-complex. This element is given explicitly as

$$j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}}) = \sum_{f\in Q}(-1)^{(|f|+1)^2} f_{\mathcal{A}}^\vee \otimes sDf + \sum_{x\in R} c_{x,\mathcal{A}}^\vee \otimes E_x \tag{23.3.6}$$

One can then apply the Connes operator $B : X(\Pi_m(\mathcal{A}) \to X(\Pi_m(\mathcal{A})$ in the X-complex (see Section 22.5) to the closed element $j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}})$. The result is automatically both b-closed and B-closed, and hence represents a negative cyclic class $[B(j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}}))] \in \mathrm{HC}^-_m(\Pi_m(\mathcal{A}))$.

Theorem 23.3.7. *The negative cyclic class $[B(j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}}))] \in \mathrm{HC}^-_m(\Pi_m(\mathcal{A}))$ is an m-Calabi-Yau structure on the m-Calabi-Yau completion $\Pi_m(\mathcal{A})$.*

Proof. Denote by $[\xi_{\mathcal{A}}] \in \mathrm{HH}_m(\Pi_m(\mathcal{A}))$ the underlying Hochschild class of $[B(j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}}))]$. We need to show that $[\xi_{\mathcal{A}}]$ is a weak m-Calabi-Yau structure.

To see this, we first apply Theorem 22.5.18 to give a representative of the Hochschild class $[\xi]$ in the double X-complex $\mathbb{X}(\Pi_m(\mathcal{A}))$. Indeed, Theorem 22.5.18 shows that $[\xi_{\mathcal{A}}]$ is represented by the closed element

$$\tilde{\xi_{\mathcal{A}}} := \tilde{B}(j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}})) \quad \in \mathbb{X}(\Pi_m(\mathcal{A}))$$

in the double X-complex. By (23.3.6), a direct calculation shows that this element is given explicitly as

$$\begin{aligned}\tilde{\xi_{\mathcal{A}}} &= \sum_{f\in Q}(-1)^{(|f|+1)^2}\big[sD(f^\vee_{\mathcal{A}})\otimes sDf + (-1)^{(m-2-|f|)(|f|+1)}sDf\otimes sD(f^\vee_{\mathcal{A}})\big] \\ &\quad + \sum_{x\in R}\big[sD(c^\vee_{x,\mathcal{A}})\otimes E_x + E_x\otimes sD(c^\vee_{x,\mathcal{A}})\big]\end{aligned} \tag{23.3.8}$$

The proof is then complete by an application of the following lemma. □

Lemma 23.3.9. *Any closed element $\tilde{\xi_{\mathcal{A}}}$ of degree m in the double complex $\mathbb{X}(\Pi_m(\mathcal{A}))$ of the form*

$$\tilde{\xi_{\mathcal{A}}} = \sum_{f\in Q}\pm\big[sD(f^\vee_{\mathcal{A}})\otimes sDf\pm sDf\otimes sD(f^\vee_{\mathcal{A}})\big]\pm\sum_{x\in R}\big[sD(c^\vee_{x,\mathcal{A}})\otimes E_x\pm E_x\otimes sD(c^\vee_{x,\mathcal{A}})\big] \tag{23.3.10}$$

gives a weak m-Calabi-Yau structure on the Calabi-Yau completion $\Pi_m(\mathcal{A})$.

Proof. The standard resolution $\operatorname{Res}(\Pi_m(\mathcal{A}))$ of $\Pi_m(\mathcal{A})$ is semi-free with basis consisting of

$$\{sDc_{x,\mathcal{A}}\}_{x\in R}\cup\{sDf^\vee_{\mathcal{A}}\}_{f\in Q}\cup\{E_x\}_{x\in R}\cup\{sDf\}_{f\in Q} \tag{23.3.11}$$

Similarly, its shifted dual $\operatorname{Res}(\Pi_m(\mathcal{A}))^\vee[m]$ is semi-free with a shifted dual basis

$$\{s^m(E_x)^\vee\}_{x\in R}\cup\{s^m(sDf)^\vee\}_{f\in Q}\cup\{s^m(sDc_{x,\mathcal{A}})^\vee\}_{x\in R}\cup\{s^m(sDf^\vee_{\mathcal{A}})^\vee\}_{f\in Q} \tag{23.3.12}$$

The closed element (23.3.10) then induces a closed map $\widehat{\tilde{\xi}_{\mathcal{A}}} : \mathrm{Res}(\Pi_m(\mathcal{A}))^\vee[m] \to \mathrm{Res}(\Pi_m(\mathcal{A}))$ of degree 0. Up to signs, this map sends the basis (23.3.11) bijectively to basis (23.3.12). This induced map is therefore an isomorphism of bimodules. □

Now, we consider the deformed case. Thus, let $\eta_{\mathcal{A}}$ be a degree $m-2$ element in the X-complex $X(\mathcal{A}) = \mathcal{A} \otimes \mathrm{Res}(\mathcal{A})$ of $\mathcal{A}$, We suppose $\eta_{\mathcal{A}}$ is given by

$$\eta_{\mathcal{A}} = \sum_{f \in Q} a_f \cdot sDf + \sum_{x \in R} a_x \cdot E_x \,,$$

then this element determines a closed map

$$\hat{\eta_{\mathcal{A}}} s^{1-m} : \mathrm{Res}(\mathcal{A})^\vee \to \mathcal{A}$$

of degree -1. We use this map to deform the differential in the Calabi-Yau completion (23.3.3), as indicated in Section 23.2.

Keeping the notation in Section 23.2, the deformed differential has the form $d = d_0 + d_1$. Moreover, we have

$$d_1(f^\vee_{\mathcal{A}}) = (-1)^{(|f|+1)^2} a_f \qquad \text{and} \qquad d_1(c_{x,\mathcal{A}}) = a_x \tag{23.3.13}$$

In the deformed case, the map (23.3.5) is no longer a chain map. In particular, the element $j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}})$ in the X-complex of $\Pi_m(\mathcal{A}; \eta_{\mathcal{A}})$ is no longer b-closed. Instead, applying the extra differential (23.3.13) to formula (23.3.6) gives

$$b(j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}})) = \sum_{f \in Q} a_f \cdot sDf + \sum_{x \in R} a_x \cdot E_x = \gamma_{\mathcal{A}}(\eta_{\mathcal{A}}) \tag{23.3.14}$$

where $\gamma_{\mathcal{A}} : X(\mathcal{A}) \to X(\Pi_n(\mathcal{A}; \eta_{\mathcal{A}}))$ is the map on the X-complexes induced by the canonical inclusion $i_{\mathcal{A}} : \mathcal{A} \to \Pi_n(\mathcal{A}; \eta_{\mathcal{A}})$.

Since $j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}})$ is not b-closed in the deformed case, we cannot simply apply Connes' operator to it to produce a closed element. The extra input needed for this case is a negative cyclic lift of the deformation parameter $\eta_{\mathcal{A}}$.

Thus, suppose that $\eta_{\mathcal{A}}$ has a lift to a power series

$$\eta_{\mathcal{A}}(u) = \eta_{\mathcal{A}}^{(0)} + u \cdot \eta_{\mathcal{A}}^{(1)} + u^2 \cdot \eta_{\mathcal{A}}^{(2)} + \ldots \in X(\mathcal{A})[[u]]$$

where $\eta_{\mathcal{A}}^{(0)} = \eta_{\mathcal{A}}$, and $\eta_{\mathcal{A}}^{(i)}$ are elements of degree $m-2+i$ in the X-complex $X(\mathcal{A})$ of $\mathcal{A}$, satisfying $B(\eta_{\mathcal{A}}^{(i)}) + b(\eta_{\mathcal{A}}^{(i+1)}) = 0$.

Let $\xi_{\mathcal{A}}(u) \in X(\Pi_n(\mathcal{A};\eta_{\mathcal{A}}))[[u]]$ be the power series in the X-complex of the deformed Calabi-Yau completion $\Pi_n(\mathcal{A};\eta_{\mathcal{A}})$, defined by

$$\xi_{\mathcal{A}}(u) := \left(B(j(s^{m-1}\theta_{\mathcal{A}})) - i_{\mathcal{A}}(\eta_{\mathcal{A}}^{(1)})\right) - u \cdot i_{\mathcal{A}}(\eta_{\mathcal{A}}^{(2)}) - u^2 \cdot i_{\mathcal{A}}(\eta_{\mathcal{A}}^{(3)}) - \ldots \tag{23.3.15}$$

then we have

Theorem 23.3.16. *The element $\xi_{\mathcal{A}}(u) \in X(\Pi_n(\mathcal{A};\eta_{\mathcal{A}}))[[u]]$ defined in* (23.3.15) *is $(b+uB)$-closed, and hence represent a class $[\xi_{\mathcal{A}}(u)] \in \mathrm{HC}_m^-(\Pi_n(\mathcal{A};\eta_{\mathcal{A}}))$ in the negative cyclic homology of $\Pi_n(\mathcal{A};\eta_{\mathcal{A}})$. Moreover, this class is an m-Calabi-Yau structure on $\Pi_n(\mathcal{A};\eta_{\mathcal{A}})$.*

Proof. The fact that $\xi_{\mathcal{A}}(u)$ is $b+uB$-closed follows from direct calculation. For example, denote by $\xi_{\mathcal{A}}$ the constant term of (23.3.15), *i.e.*,

$$\xi_{\mathcal{A}} = \xi_{\mathcal{A}}^{(0)} := B(j(s^{m-1}\theta_{\mathcal{A}})) - i_{\mathcal{A}}(\eta_{\mathcal{A}}^{(1)}) \tag{23.3.17}$$

then the calculation

$$b(\xi_{\mathcal{A}}) = -B\left(b(j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}}))\right) - b\left(i_{\mathcal{A}}(\eta_{\mathcal{A}}^{(1)})\right) \overset{(23.3.14)}{=} -B\left(i_{\mathcal{A}}(\eta_{\mathcal{A}}^{(0)})\right) - b\left(i_{\mathcal{A}}(\eta_{\mathcal{A}}^{(1)})\right) = 0$$

shows that $b+uB(\xi_{\mathcal{A}}(u))$ has trivial constant term. The fact that $b+uB(\xi_{\mathcal{A}}(u))$ has trivial higher terms follows from the fact that $\eta_{\mathcal{A}}(u)$ is itself $(b+uB)$-closed.

The underlying Hochschild class of $[\xi_{\mathcal{A}}(u)]$ is represented by the element (23.3.15) in the X-complex. We now find a lift $\tilde{\xi_{\mathcal{A}}}$ of this element to a closed element in the double X-complex. In view of the formula (23.3.17), we expect such a lift to be of a form

$$\tilde{\xi_{\mathcal{A}}} = \tilde{B}(j(s^{m-1}\theta_{\mathcal{A}})) - i_{\mathcal{A}}(\tilde{\eta_{\mathcal{A}}}^{(1)}) \tag{23.3.18}$$

where $\tilde{B} : X(\Pi_n(\mathcal{A};\eta_{\mathcal{A}})) \to \mathbb{X}(\Pi_n(\mathcal{A};\eta_{\mathcal{A}}))$ is the lifted Connes operator and $\tilde{\eta_{\mathcal{A}}}^{(1)} \in \mathbb{X}(\mathcal{A})$ is an element in the double X-complex of $\mathcal{A}$.

We want the element (23.3.18) to be b-closed. A direct calculation shows that

$$\begin{aligned} b(\tilde{\xi_{\mathcal{A}}}) &= -\tilde{B}b(j(s^{m-1}\theta_{\mathcal{A}})) - b(\gamma_{\mathcal{A}}(\tilde{\eta_{\mathcal{A}}}^{(1)})) \\ &\overset{(23.3.14)}{=} -\tilde{B}(\gamma_{\mathcal{A}}(\eta_{\mathcal{A}}^{(0)}) - i_{\mathcal{A}}(b(\tilde{\eta_{\mathcal{A}}}^{(1)})) \\ &= \gamma_{\mathcal{A}}(-\tilde{B}(\eta_{\mathcal{A}}^{(0)}) - b(\tilde{\eta_{\mathcal{A}}}^{(1)})) \end{aligned}$$

Thus, it suffices to find an element $\tilde{\eta_{\mathcal{A}}}^{(1)} \in \mathbb{X}(\mathcal{A})$ in the double X-complex of $\mathcal{A}$ such that

$$\tilde{B}(\eta_{\mathcal{A}}^{(0)}) + b(\tilde{\eta_{\mathcal{A}}}^{(1)}) = 0 \quad \text{in } \mathbb{X}(\mathcal{A}), \qquad \text{and} \qquad p(\tilde{\eta_{\mathcal{A}}}^{(1)}) = \eta_{\mathcal{A}}^{(1)} \in X(\mathcal{A}). \tag{23.3.19}$$

First, notice that $\tilde{B}(\eta_{\mathcal{A}}^{(0)})$ projects to $B(\eta_{\mathcal{A}}^{(0)})$ under the weak equivalence $p : \mathbb{X}(\mathcal{A}) \to X(\mathcal{A})$. Since $B(\eta_{\mathcal{A}}^{(0)})$ is b-exact, so is $\tilde{B}(\eta_{\mathcal{A}}^{(0)})$. Therefore, there exists an element $\tilde{\eta_{\mathcal{A}}}' \in \mathbb{X}(\mathcal{A})$ such that $\tilde{B}(\eta_{\mathcal{A}}^{(0)}) + b(\tilde{\eta_{\mathcal{A}}}') = 0$.

While the projection $p(\tilde{\eta_{\mathcal{A}}}')$ may not be equal to $\eta_{\mathcal{A}}^{(1)}$, their difference is nonetheless b-closed. Indeed, we have

$$b(p(\tilde{\eta_{\mathcal{A}}}')) = p(b(\tilde{\eta_{\mathcal{A}}}')) = -p(\tilde{B}(\eta_{\mathcal{A}}^{(0)})) = -B(\eta_{\mathcal{A}}^{(0)}) = b(\eta_{\mathcal{A}}^{(1)})$$

Now, the projection map $p : \mathbb{X}(\mathcal{A}) \to X(\mathcal{A})$ is a surjective quasi-isomorphism. Therefore, it induces a surjective map on closed elements

$\mathrm{Z}_\bullet(\mathbb{X}(\mathcal{A})) \to \mathrm{Z}_\bullet(X(\mathcal{A}))$. In particular, the b-closed element $\eta^{(1)}_{\mathcal{A}} - p(\tilde{\eta_{\mathcal{A}}}') \in X(\mathcal{A})$ lifts to a b-closed element $\tilde{\eta_{\mathcal{A}}}'' \in \mathbb{X}(\mathcal{A})$.

Define the element $\tilde{\eta_{\mathcal{A}}}^{(1)} \in \mathbb{X}(\mathcal{A})$ by

$$\tilde{\eta_{\mathcal{A}}}^{(1)} := \tilde{\eta_{\mathcal{A}}}' + \tilde{\eta_{\mathcal{A}}}''$$

Then, by construction, this element satisfies the conditions (23.3.19). Therefore, the element (23.3.18) represents the underlying Hochschild class of the negative cyclic class $[\xi_{\mathcal{A}}(u)] \in \mathrm{HC}^-_m(\Pi_n(\mathcal{A};\eta_{\mathcal{A}}))$ in the theorem. To see that this Hochschild class defines a weak Calabi-Yau structure, we notice that, by the calculation (23.3.8), the element (23.3.18) has the form

$$\begin{aligned}\tilde{\xi_{\mathcal{A}}} &= \sum_{f\in Q}(-1)^{(|f|+1)^2}\big[sD(f^\vee_{\mathcal{A}})\otimes sDf + (-1)^{(m-2-|f|)(|f|+1)}sDf\otimes sD(f^\vee_{\mathcal{A}})\big]\\ &\quad+\sum_{x\in R}\big[sD(c^\vee_{x,\mathcal{A}})\otimes E_x + E_x\otimes sD(c^\vee_{x,\mathcal{A}})\big] \;+\; \gamma_{\mathcal{A}}(\tilde{\eta_{\mathcal{A}}}^{(1)})\end{aligned}$$

The proof of the theorem is then complete by an application of the following lemma, which generalizes Lemma 23.3.9. □

Lemma 23.3.20. *Suppose that $\tilde{\xi_{\mathcal{A}}} \in \mathbb{X}(\Pi_m(\mathcal{A};\eta_{\mathcal{A}}))$ is a closed element of degree m of the form*

$$\sum_{f\in Q}\pm\big[sD(f^\vee_{\mathcal{A}})\otimes sDf\pm sDf\otimes sD(f^\vee_{\mathcal{A}})\big]\pm\sum_{x\in R}\big[sD(c^\vee_{x,\mathcal{A}})\otimes E_x\pm E_x\otimes sD(c^\vee_{x,\mathcal{A}})\big]+\gamma_{\mathcal{A}}(\tilde{\eta_{\mathcal{A}}}^{(1)})$$

where $\tilde{\eta_{\mathcal{A}}}^{(1)} \in \mathbb{X}(\mathcal{A})$ is an element in the double X-complex of $\mathcal{A}$, then $\tilde{\xi_{\mathcal{A}}}$ gives a weak m-Calabi-Yau structure on the deformed Calabi-Yau completion $\Pi_m(\mathcal{A};\eta_{\mathcal{A}})$.

Proof. The underlying graded k-category of the deformed m-Calabi-Yau completion $\Pi_m(\mathcal{A};\eta_{\mathcal{A}})$ is the same as that of the undeformed one. Therefore, the standard resolution $\mathrm{Res}(\Pi_m(\mathcal{A};\eta_{\mathcal{A}}))$ of $\Pi_m(\mathcal{A};\eta_{\mathcal{A}})$ is still semi-free with the same basis (23.3.11). Hence, its shifted dual also has the basis (23.3.12).

The closed element $\tilde{\xi_{\mathcal{A}}}$ induces a closed map $\tilde{\xi_{\mathcal{A}}} s^{-m} : \mathrm{Res}(\Pi_m(\mathcal{A};\eta_{\mathcal{A}}))^\vee[m] \to \mathrm{Res}(\Pi_m(\mathcal{A};\eta_{\mathcal{A}}))$ of degree zero. With respect to the basis (23.3.11) and (23.3.12), this map is given by the matrix

$$\begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ * & * & \pm 1 & 0 \\ * & * & 0 & \pm 1 \end{bmatrix}$$

This map is therefore an isomorphism of bimodules. □

Remark 23.3.21. Our proof shows that, if the deformation parameter $[\eta_{\mathcal{A}}] \in \mathrm{HC}^-_{m-2}(\mathcal{A})$ can be represented by an element $\eta_{\mathcal{A}} \in X(\mathcal{A})$ that is both b-closed and B-closed, (for example, this is the case if $[\eta_{\mathcal{A}}]$ is in the image of $B : \mathrm{HC}_{m-3}(\mathcal{A}) \to \mathrm{HC}^-_{m-2}(\mathcal{A})$,) then the element $j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}})$ in the X-complex of the deformed Calabi-Yau completion is b-closed. Therefore, the canonical m-Calabi-Yau structure on the deformed Calabi-Yau completion is then B-exact. Namely, it is the image of $[j_{\mathcal{A}}(s^{m-1}\theta_{\mathcal{A}})] \in \mathrm{HH}_{m-1}(\Pi_m(\mathcal{A};\eta_\eta))$ under the Connes map $B : \mathrm{HH}_{m-1}(\Pi_m(\mathcal{A};\eta_\eta)) \to \mathrm{HC}^-_m(\Pi_m(\mathcal{A};\eta_\eta))$. See [48, 162] for a discussion of the exactness condition.

Now we give a counter-example to [93, Theorem 5.2]. This shows that the existence of negative cyclic lift $[\eta_{\mathcal{A}}(u)]$ of the deformation parameter $[\eta_{\mathcal{A}}]$ is necessary even if one is primarily interested in weak Calabi-Yau structures.

Let $A = k\langle x\rangle$ be the free associative algebra on one varaible x, considered as a DG category with one object 0. Consider the deformation parameter $[x] \in A = A_\natural = \mathrm{HH}_0(A)$. Notice that $B([x])$ is nonzero in $\mathrm{HH}_1(A)$. Therefore, $[x]$ does not have any negative cyclic lift. We claim that the deformed 2-Calabi-Yau completion $\Pi := \Pi_2(A;[z])$ with respect to the deformation parameter $[x]$ does not have a weak 2-Calabi-Yau structure.

A direct calculation shows that Π is given by

$$\Pi = k\langle x, y, t\rangle, \qquad |x| = |y| = 0, \qquad |t| = 1, \qquad d(t) = xy - yx + x$$

Its standard resolution $\mathrm{Res}(\Pi)$ has a basis given by $\{E_0,\, sDx,\, sDy,\, sDt\}$, with differentials

$$\begin{aligned} d(E_0) &= 0 \qquad d(sDx) = x \cdot E_0 - E_0 \cdot x \qquad d(sDy) = y \cdot E_0 - E_0 \cdot y \\ d(sDt) &= t \cdot E_0 - E_0 \cdot t - sD(xy - yx + x) \\ &= t \cdot E_0 - E_0 \cdot t - sDx \cdot y - x \cdot sDy + sDy \cdot x + y \cdot sDx - sDx \end{aligned} \tag{23.3.22}$$

Therefore, the chain complex $\Pi \otimes_{\Pi^e} \mathrm{Res}(\Pi)$ is given by the following diagram

$$\begin{tikzcd} & \Pi \cdot sDx & \\ \Pi \cdot sDt \arrow[ur, "d"] & & \Pi \cdot E_0 \\ & \Pi \cdot sDy & \end{tikzcd}$$

where the only non-trivial differential is from the component $\Pi \cdot sDt$ to the component $\Pi \cdot sDx$, given by $sDt \mapsto -sDx$. Thus, its homology is given by $\mathrm{HH}_\bullet(\Pi;\Pi) = \mathrm{H}_\bullet(\Pi) \oplus \mathrm{H}_\bullet(\Pi)[1]$.

On the other hand, the dual $\mathrm{Res}(\Pi)^\vee$ of the standard resolution of Π has basis

$$\{(E_0)^\vee,\, (sDx)^\vee,\, (sDy)^\vee,\, (sDt)^\vee\},$$

with differentials

$$\begin{aligned} &d(sDt)^\vee = 0 \qquad d(sDy)^\vee = -x \cdot (sDt)^\vee + (sDt)^\vee \cdot x \\ &d(sDx)^\vee = y \cdot (sDt)^\vee - (sDt)^\vee \cdot y + (sDt)^\vee \\ &d(E_0)^\vee = -(sDx)^\vee \cdot x + x \cdot (sDx)^\vee - (sDy)^\vee \cdot y + y \cdot (sDy)^\vee - (E_0)^\vee \cdot t + t \cdot (E_0)^\vee \end{aligned} \tag{23.3.23}$$

Thus, the chain complex $\Pi \otimes_{\Pi^e} (\mathrm{Res}(\Pi))^\vee[2]$ is given by the following diagram

$$\begin{tikzcd}[row sep=small] & \Pi \cdot s^2(sDy)^\vee & \\ \Pi \cdot s^2(E_0)^\vee & & \Pi \cdot s^2(sDt)^\vee \\ & \Pi \cdot s^2(sDx)^\vee \arrow[ur, "d"] & \end{tikzcd}$$

where the only non-trivial differential is from the component $\Pi \cdot s^2(sDx)^\vee$ to the component $\Pi \cdot s^2(sDt)^\vee$, given by $s^2(sDx)^\vee \mapsto s^2(sDt)^\vee$. Thus, its homology is given by $\mathrm{HH}_\bullet(\Pi; \Pi^![2]) = \mathrm{H}_\bullet(\Pi)[1] \oplus \mathrm{H}_\bullet(\Pi)[2]$.

Notice that $\mathrm{H}_0(\Pi)$ is nonzero because its abelianization is simply $k[y]$, which is nonzero. Therefore, $\mathrm{HH}_0(\Pi; \Pi)$ is nonzero, but $\mathrm{HH}_0(\Pi; \Pi^![2])$ is zero. The bimodules $\Pi^![2]$ and Π are therefore not isomorphic in the derived category $\mathcal{D}(\Pi^e)$.

23.4 The relative case

We extend the proof in the previous subsection to the relative case. Most of the proof is completely parallel to the absolute case. We will therefore work directly with the deformed case.

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a DG functor between k-flat DG categories of fintie Sullivan type. Since relative Calabi-Yau completion is invariant under quasi-equivalences between k-flat categories (see Section 23.2), to prove Theorem 23.2.18, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are finitely Sullivan. Moreover, by passing to the mapping cylinder of F (see Remark 24.2.8 below), we can assume that F is a semi-free extension.

Thus, there is a finite graded quiver (R_2, Q_2) and a graded subquiver $(R_1, Q_1) \subset (R_2, Q_2)$, such that $\mathcal{A} = T_{R_1}(Q_1)$ and $\mathcal{B} = T_{R_2}(Q_2)$ as graded k-categories.

The dual $\mathrm{Res}(\mathcal{B})^\vee$ of standard bimodule resolution $\mathrm{Res}(\mathcal{B})$ of $\mathcal{B}$ is semi-free with basis

$$\{(sDf_{\mathcal{B}})^\vee\}_{f\in Q_2} \cup \{(E_{x,\mathcal{B}})^\vee\}_{x\in R_2}$$

Therefore, the bimodule $\Xi = \mathrm{cone}\left[\mathrm{Res}(\mathcal{B})^\vee \xrightarrow{i^\vee} F_!\mathrm{Res}(\mathcal{A})^\vee\right][n-2]$ is semi-free with basis

$$\{s^{n-1}(sDf_{\mathcal{B}})^\vee\}_{f\in Q_2} \cup \{s^{n-1}(E_{x,\mathcal{B}})^\vee\}_{x\in R_2} \cup \{s^{n-2}(sDf_{\mathcal{A}})^\vee\}_{f\in Q_1} \cup \{s^{n-2}(E_{x,\mathcal{A}})^\vee\}_{x\in R_1}$$

Rewrite these elements as

$$\begin{aligned} f_{\mathcal{B}}^\vee &:= s^{n-1}(sDf_{\mathcal{B}})^\vee \qquad & c_{x,\mathcal{B}} &:= s^{n-1}(E_{x,\mathcal{B}})^\vee \\ f_{\mathcal{A}}^\vee &:= s^{n-2}(sDf_{\mathcal{A}})^\vee \qquad & c_{x,\mathcal{A}} &:= s^{n-2}(E_{x,\mathcal{A}})^\vee \end{aligned} \tag{23.4.1}$$

The generators $f_{\mathcal{B}}^\vee$ and $f_{\mathcal{A}}^\vee$ have degrees given by $|f_{\mathcal{B}}^\vee| = n-2-|f|$ and $|f_{\mathcal{A}}^\vee| = n-3-|f|$, pointing in the opposite direction to f. The generators $c_{x,\mathcal{B}}$ and $c_{x,\mathcal{A}}$ have degrees given by $|c_{x,\mathcal{B}}| = n-1$ and $|c_{x,\mathcal{A}}| = n-2$, pointing from x to x.

The (deformed) relative Calabi-Yau completion is then given by

$$\begin{tikzcd} \Pi_{n-1}(\mathcal{A};\eta_{\mathcal{A}}) \arrow[r, equal] \arrow[d, "\tilde{F}"] & T_{R_1}(\{f\}_{f\in Q_1} \cup \{f_{\mathcal{A}}^\vee\}_{f\in Q_1} \cup \{c_{x,\mathcal{A}}\}_{x\in R_1}) \arrow[d] \\ \Pi_n(\mathcal{B},\mathcal{A};\eta) \arrow[r, equal] & T_{R_2}(\{f\}_{f\in Q_1} \cup \{f_{\mathcal{A}}^\vee\}_{f\in Q_1} \cup \{c_{x,\mathcal{A}}\}_{x\in R_1} \cup \{f_{\mathcal{B}}^\vee\}_{f\in Q_2} \cup \{c_{x,\mathcal{B}}\}_{x\in R_2}) \end{tikzcd} \tag{23.4.2}$$

where $\eta = (s\eta_{\mathcal{A}}, \eta_{\mathcal{B}}) \in \mathrm{cone}\big[X(\mathcal{A}) \to X(\mathcal{B})\big]$ is a closed element of degree $n-2$ used to deform the differentials in the relative Calabi-Yau completion (see Section 23.2).

For brevity of notation, we will denote the deformed relative Calabi-Yau completion simply as

$$\Pi_{\mathcal{A}} := \Pi_{n-1}(\mathcal{A};\eta_{\mathcal{A}}) \qquad \text{and} \qquad \Pi_{\mathcal{B}} := \Pi_n(\mathcal{B},\mathcal{A};\eta)$$

We wish to construct a relative Calabi-Yau structure on the map $\tilde{F}$: $\Pi_{n-1}(\mathcal{A};\eta_{\mathcal{A}}) \to \Pi_n(\mathcal{B},\mathcal{A};\eta)$. As in the absolute case, there are two ingredients. The first is given by Casimir elements.

Thus, consider the elements $\theta_{\mathcal{A}} \in \mathrm{Res}(\mathcal{A})^\vee \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{A})$ and $\theta_{\mathcal{B}} \in \mathrm{Res}(\mathcal{B})^\vee \otimes_{\mathcal{B}^e} \mathrm{Res}(\mathcal{B})$ defined respectively by the equations

$$\theta_{\mathcal{A}} \;=\; \sum_{f\in Q_1} (-1)^{(|f|+1)^2} (sDf_{\mathcal{A}})^\vee \otimes sDf \;+\; \sum_{x\in R_1} (E_{x,\mathcal{A}})^\vee \otimes E_x \tag{23.4.3}$$

$$\theta_{\mathcal{B}} \;=\; \sum_{f\in Q_2} (-1)^{(|f|+1)^2} (sDf_{\mathcal{B}})^\vee \otimes sDf \;+\; \sum_{x\in R_2} (E_{x,\mathcal{B}})^\vee \otimes E_x \tag{23.4.4}$$

then both of closed elements of degree zero, as they represent the identity maps on $\mathrm{Res}(\mathcal{A})^\vee$ and $\mathrm{Res}(\mathcal{B})^\vee$ respectively.

These two elements are related in the following way. Consider the maps

$$\mathrm{Res}(\mathcal{A})^\vee \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{A}) \xrightarrow{\mathrm{id}\otimes\gamma_F} \mathrm{Res}(\mathcal{A})^\vee \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{B}) = F_!(\mathrm{Res}(\mathcal{A})^\vee) \otimes_{\mathcal{B}^e} \mathrm{Res}(\mathcal{B}) \xleftarrow{\gamma_F^\vee\otimes\mathrm{id}} \mathrm{Res}(\mathcal{B})^\vee \otimes_{\mathcal{B}^e} \mathrm{Res}$$

then we have

$$(\mathrm{id}\otimes\gamma_F)(\theta_{\mathcal{A}}) = (\gamma_F^\vee \otimes \mathrm{id})(\theta_{\mathcal{B}}) \tag{23.4.5}$$

since both elements correspond to the map $\gamma_F^\vee : \mathrm{Res}(\mathcal{B})^\vee \to F_!(\mathrm{Res}(\mathcal{A})^\vee)$.

Now, consider the cone

$$\Theta(\mathcal{B},\mathcal{A}) \;:=\; \mathrm{cone}\left[\; \mathrm{Res}(\mathcal{A})^\vee[n-2] \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{A}) \xrightarrow{\pm\iota\otimes\gamma_F} \Xi \otimes_{\mathcal{B}^e} \mathrm{Res}(\mathcal{B}) \;\right] \tag{23.4.6}$$

where we have written $\pm\iota\otimes\gamma_F := ((-1)^{n-2} s^{n-2}\iota s^{2-n}) \otimes \gamma_F$ for brevity of notation.

This cone can be written as

$$\Theta(\mathcal{B},\mathcal{A}) = \left[\begin{tikzcd} & \mathrm{Res}(\mathcal{B})^\vee[n-1] \otimes_{\mathcal{B}^e} \mathrm{Res}(\mathcal{B}) \arrow[d, "(-1)^{n-2}s^{n-2}\gamma_F^\vee s^{1-n}\otimes\mathrm{id}"] \\ \mathrm{Res}(\mathcal{A})^\vee[n-1] \otimes_{\mathcal{A}^e} \mathrm{Res}(\mathcal{A}) \arrow[r, "(-1)^{n-2}s^{-1}\otimes\gamma_F"] & F_!\mathrm{Res}(\mathcal{A})^\vee[n-2] \otimes_{\mathcal{B}^e} \mathrm{Res}(\mathcal{B}) \end{tikzcd}\right]$$

Thus, if we consider the pair of elements

$$(-s^{n-1}\theta_{\mathcal{A}}\,,\, s^{n-1}\theta_{\mathcal{B}}) \;\in\; \big(\operatorname{Res}(\mathcal{A})^\vee[n{-}1]\otimes_{\mathcal{A}^e}\operatorname{Res}(\mathcal{A})\big) \oplus \big(\operatorname{Res}(\mathcal{B})^\vee[n{-}1]\otimes_{\mathcal{B}^e}\operatorname{Res}(\mathcal{B})\big)$$

to be an element in $\Theta(\mathcal{B},\mathcal{A})$, then this element is closed of degree $n-1$. We call this the *relative Casimir element*, and denote it as

$$s^{n-1}\theta_{\mathcal{B},\mathcal{A}} \in \Theta(\mathcal{B},\mathcal{A}) \tag{23.4.7}$$

The second ingredient in the construction of a relative Calabi-Yau structure is a map

$$j : \Theta(\mathcal{B},\mathcal{A}) \;\to\; X(\Pi_{\mathcal{B}},\Pi_{\mathcal{A}}) := \operatorname{cone}\big[X(\Pi_{\mathcal{A}}) \xrightarrow{\gamma_{\tilde{F}}} X(\Pi_{\mathcal{B}})\big] \tag{23.4.8}$$

The map is constructed in the same way as in the absolute case. Namely, consider the following commutative diagram

$$\begin{tikzcd} \operatorname{Res}(\mathcal{A})^\vee[n-2]\otimes_{\mathcal{A}^e}\operatorname{Res}(\mathcal{A}) \arrow[r, "j_{\mathcal{A}}"] \arrow[d, "\pm\,\iota\otimes\gamma_F"] & \Pi_{\mathcal{A}}\otimes_{\Pi_{\mathcal{A}}^e}\operatorname{Res}(\Pi_{\mathcal{A}}) \arrow[d, "\gamma_{\tilde{F}}"] \\ \Xi\otimes_{\mathcal{B}^e}\operatorname{Res}(\mathcal{B}) \arrow[r, "j_{\mathcal{B}}"] & \Pi_{\mathcal{B}}\otimes_{\Pi_{\mathcal{B}}^e}\operatorname{Res}(\Pi_{\mathcal{B}}) \end{tikzcd} \tag{23.4.9}$$

where the map $\pm\,\iota\otimes\gamma_F$ is as in (23.4.6). While the horizontal maps $j_{\mathcal{A}}$ and $j_{\mathcal{B}}$ are not chain maps, they nonetheless assemble to give a map on the cones. This defines the map (23.4.8).

The image $j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}})$ of the relative Casimir element (23.4.7) under the map (23.4.8) is then given by the pair of elements

$$\big(-sj_{\mathcal{A}}(s^{n-2}\theta_{\mathcal{A}})\,,\, j_{\mathcal{B}}(s^{n-1}\theta_{\mathcal{B}})\big) \;\in\; X(\Pi_{\mathcal{A}})[1] \,\oplus\, X(\Pi_{\mathcal{B}})$$

By a direct calculation, these elements are given by

$$\begin{aligned} -sj_{\mathcal{A}}(s^{n-2}\theta_{\mathcal{A}}) &= -s\left(\sum_{f\in Q_1}(-1)^{(|f|+1)^2} f_{\mathcal{A}}^\vee\otimes sDf + \sum_{x\in R_1} c_{x,\mathcal{A}}\otimes E_x\right) \\ j_{\mathcal{B}}(s^{n-1}\theta_{\mathcal{B}}) &= \sum_{f\in Q_2}(-1)^{(|f|+1)^2} f_{\mathcal{B}}^\vee\otimes Df + \sum_{x\in R_2} c_{x,\mathcal{B}}\otimes E_x \end{aligned} \tag{23.4.10}$$

Lemma 23.4.11. *The image $j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}})$ of the relative Casimir element* (23.4.7) *under the map* (23.4.8) *has b-differential given by*

$$b(j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}})) = \gamma(\eta)$$

where $\gamma(\eta)$ is the iamge of the deformation parameter $\eta = (s\eta_{\mathcal{A}}, \eta_{\mathcal{B}}) \in X(\mathcal{B}, \mathcal{A})$ under the canonical map

$$\gamma : X(\mathcal{B}, \mathcal{A}) \to X(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$$

of relative X-complexes.

Proof. In the undeformed case, the maps $j_{\mathcal{A}}$ and $j_{\mathcal{B}}$ in the diagram (23.4.9) commute with differential. Hence so does the induced map (23.4.8) on the cone. Since $\theta_{\mathcal{B},\mathcal{A}}$ is closed in $\Theta(\mathcal{B}, \mathcal{A})$, so is the image of its shift under (23.4.8). *i.e.*, we have $b(j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}})) = 0$ in this case.

In the deformed case, the differentials on $\Pi_{\mathcal{A}}$ and $\Pi_{\mathcal{B}}$ have two parts $d = d_0 + d_1$, where d_1 come from the deformation parameter $\eta = (s\eta_{\mathcal{A}}, \eta_{\mathcal{B}})$. This induces a decomposition of the differential on the relative X-complex $X(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ into three components $b = b_0 + b_1 + b_2$, where

$$b_0 = d_0 \otimes \mathrm{id} + \mathrm{id} \otimes d_0 \qquad\qquad b_1 = d_1 \otimes \mathrm{id} + \mathrm{id} \otimes d_1$$

and b_2 is the extra differential given by the map $\gamma_{\tilde{F}}$ of X-complexes defining $X(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ as a cone (see (23.4.8)).

Thus, the differential $b_0 + b_2$ is precisely the differential of $X(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ for the undeformed relative Calabi-Yau completion. By the above argument, we have $(b_0 + b_2)(j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}})) = 0$. Hence, $b(j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}})) = b_1(j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}}))$. A direct calculation analogous to the absolute case then shows that this is equal to $i(\eta)$. □

As in the absolute case, we now use the fact that $\eta \in X(\mathcal{B}, \mathcal{A})$ has a negative cyclic lift

$$\eta(u) = \eta^{(0)} \,+\, u \cdot \eta^{(1)} \,+\, u^2 \cdot \eta^{(2)} \,+\, \dots \tag{23.4.12}$$

for elements $\eta^{(i)} \in X(\mathcal{B}, \mathcal{A})$ such that $\eta^{(0)} = \eta$ and $B(\eta^{(i)}) + b(\eta^{(i+1)}) = 0$. Then define the power series $\xi(u) \in X(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})[[u]]$ in the relative X-complex of the deformed relative Calabi-Yau completion by

$$\xi(u) \,:=\, \big(B(j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}})) - \gamma(\eta^{(1)})\big) \,-\, u \cdot \gamma(\eta^{(2)}) \,-\, u^2 \cdot \gamma(\eta^{(3)}) \,-\, \dots \tag{23.4.13}$$

Then we have

Theorem 23.4.14. *The element $\xi(u) \in X(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})[[u]]$ defined in* (23.4.13) *is $(b{+}uB)$-closed, and hence represent a cloass $[\xi(u)] \in \mathrm{HC}^{-}_{m}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ in the relative negative cyclic homology. Moreover, this class is a relative n-Calabi-Yau structure on the relative deformed Calabi-Yau completion.*

Proof. As in the absolute case, one can find an element $\tilde{\eta}^{(1)} \in \mathbb{X}(\mathcal{B}, \mathcal{A})$ in the relative double X-complex such that

$$\tilde{B}(\eta^{(0)}) + b(\tilde{\eta}^{(1)}) = 0 \quad \text{in } \mathbb{X}(\mathcal{B}, \mathcal{A}), \qquad \text{and} \qquad p(\tilde{\eta}^{(1)}) = \eta^{(1)} \in X(\mathcal{B}, \mathcal{A})$$

Then the element

$$\tilde{\xi} = \tilde{B}(j(s^{n-1}\theta_{\mathcal{B},\mathcal{A}})) - \gamma(\tilde{\eta}^{(1)}) \quad \in \mathbb{X}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$$

represents the underlying Hochschild class of $[\xi(u)]$. A direct calculation then shows that the element $\tilde{\xi}$ can be written as

$$\tilde{\xi} = (s\xi_{\mathcal{A}}, \xi_{\mathcal{B}}) \,\in \mathbb{X}(\Pi_{\mathcal{A}})[1] \oplus \mathbb{X}(\Pi_{\mathcal{B}}) = \mathrm{cone}\left[\mathbb{X}(\Pi_{\mathcal{A}}) \xrightarrow{\gamma_{\tilde{F}}} \mathbb{X}(\Pi_{\mathcal{B}})\right]$$

where the elements $\xi_{\mathcal{A}}$ and $\xi_{\mathcal{B}}$ are given respectively by

$$\begin{aligned}\tilde{\xi_{\mathcal{A}}} \;=\; &\sum_{f\in Q_1}(-1)^{(|f|+1)^2}\big[sD(f^{\vee}_{\mathcal{A}})\otimes sDf + (-1)^{(n-3-|f|)(|f|+1)}sDf\otimes sD(f^{\vee}_{\mathcal{A}})\big]\\ &+\sum_{x\in R_1}\big[sD(c^{\vee}_{x,\mathcal{A}})\otimes E_x + E_x\otimes sD(c^{\vee}_{x,\mathcal{A}})\big] \;+\; \gamma_{\mathcal{A}}(\tilde{\eta_{\mathcal{A}}}^{(1)})\end{aligned}$$

$$\begin{aligned}\tilde{\xi_{\mathcal{B}}} \;=\; &\sum_{f\in Q_2}(-1)^{(|f|+1)^2}\big[sD(f^{\vee}_{\mathcal{B}})\otimes sDf + (-1)^{(n-2-|f|)(|f|+1)}sDf\otimes sD(f^{\vee}_{\mathcal{B}})\big]\\ &+\sum_{x\in R_2}\big[sD(c^{\vee}_{x,\mathcal{B}})\otimes E_x + E_x\otimes sD(c^{\vee}_{x,\mathcal{B}})\big] \;+\; \gamma_{\mathcal{B}}(\tilde{\eta_{\mathcal{B}}}^{(1)})\end{aligned}$$

The proof in then finished by an application of the following lemma. $\square$

Lemma 23.4.15. *Suppose that $\tilde{\xi} = (s\tilde{\xi_{\mathcal{A}}}, \tilde{\xi_{\mathcal{B}}}) \in \mathbb{X}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ is a closed element of degree n, where the elements $\tilde{\xi_{\mathcal{A}}} \in \mathbb{X}(\Pi_{\mathcal{A}})$ and $\tilde{\xi_{\mathcal{B}}} \in \mathbb{X}(\Pi_{\mathcal{B}})$ are of the form*

$$\tilde{\xi_{\mathcal{A}}} = \sum_{f\in Q_1}\pm\big[sD(f^{\vee}_{\mathcal{A}})\otimes sDf\pm sDf\otimes sD(f^{\vee}_{\mathcal{A}})\big]\pm\sum_{x\in R_1}\big[sD(c^{\vee}_{x,\mathcal{A}})\otimes E_x\pm E_x\otimes sD(c^{\vee}_{x,\mathcal{A}})\big]+\gamma_{\mathcal{A}}(\tilde{\eta_{\mathcal{A}}}^{(1)})$$

$$\tilde{\xi_{\mathcal{B}}} = \sum_{f\in Q_2}\pm\big[sD(f^{\vee}_{\mathcal{B}})\otimes sDf\pm sDf\otimes sD(f^{\vee}_{\mathcal{B}})\big]\pm\sum_{x\in R_2}\big[sD(c^{\vee}_{x,\mathcal{B}})\otimes E_x\pm E_x\otimes sD(c^{\vee}_{x,\mathcal{B}})\big]+\gamma_{\mathcal{B}}(\tilde{\eta_{\mathcal{B}}}^{(1)})$$

where $\tilde{\eta_{\mathcal{A}}}^{(1)} \in \mathbb{X}(\mathcal{A})$ and $\tilde{\eta_{\mathcal{B}}}^{(1)} \in \mathbb{X}(\mathcal{B})$ are elements in the double X-complexes of $\mathcal{A}$ and $\mathcal{B}$ respectively, then $\tilde{\xi}$ gives a weak relative n-Calabi-Yau structure on the deformed relative Calabi-Yau completion $\tilde{F} : \Pi_{\mathcal{A}} \to \Pi_{\mathcal{B}}$.

The proof of this lemma is more involved than in the absolute case. We use the homological perturbation lemma from [106, 32]. We recall the lemma, written in way that suits our need.

Lemma 23.4.16. *Given a (right) DG module of the form $M \oplus M'$ over a DG category $\mathcal{A}$, such that*

(1) Both $M_\bullet$ and $M'_\bullet$ are graded submodules when $M_\bullet \oplus M'_\bullet$ is considered as a graded module over the graded k-category $\mathcal{A}$.

(2) The graded submodule $M'_\bullet$ is itself a complex. i.e., if we write the differential of $M_\bullet \oplus M'_\bullet$ as

$$d = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : M_n \oplus M'_n \to M_{n-1} \oplus M'_{n-1}$$

then we have $\delta^2 = 0$ (or equivalently, $\gamma\beta = 0$).

(3) The submodule $M'_\bullet$ is $\mathcal{A}$-linearly contractible, i.e., there is a grade $\mathcal{A}$-linear map $M'_\bullet \to M'_\bullet$ of degree 1 such that $H\delta + \delta H = \mathrm{id}$.

Then the map

$$\alpha - \beta H \gamma : M \to M$$

defines a differential on M that extends the underlying graded $\mathcal{A}$-module structure of M to a DG module over $\mathcal{A}$.

Moreover, if we define the maps

$$\begin{aligned} i : M &\to M \oplus M', & x &\mapsto (x, -H\gamma(x)) \\ p : M \oplus M' &\to M, & (x, x') &\mapsto x - \beta H(x') \end{aligned} \tag{23.4.17}$$

then these maps are quasi-isomorphism of DG modules.

Proof of Lemma 23.4.15. We need to show that the induced map

$$\tilde{\xi}_{\mathcal{A}} s^{1-n} : \mathrm{Res}(\Pi_{\mathcal{A}})^\vee[n-1] \to \mathrm{Res}(\Pi_{\mathcal{A}}) \tag{23.4.18}$$

is a quasi-isomorphism of $\Pi_{\mathcal{A}}$-bimodules, and the induced map

$$\tilde{\xi}_{\mathcal{B}} s^{-n} : \mathrm{Res}(\Pi_{\mathcal{B}})^\vee[n] \to \mathrm{Res}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}}) := \mathrm{cone}\left[\tilde{F}_! \mathrm{Res}(\Pi_{\mathcal{A}}) \xrightarrow{\gamma_{\tilde{F}}} \mathrm{Res}(\Pi_{\mathcal{B}})\right] \tag{23.4.19}$$

is a quasi-isomorphism of $\Pi_{\mathcal{B}}$-bimodules.

The fact that (23.4.18) is a quasi-isomorphism is precisely the content of Lemma 23.3.20. Thus it suffices to show that (23.4.19) is a quasi-isomorphism. We will show this by applying Lemma 23.4.16 to the bimodule $\mathrm{Res}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ defined in (23.4.19).

First, by the explicit presentation (23.4.2) of $\Pi_{\mathcal{B}}$, the standard bimodule resolution $\mathrm{Res}(\Pi_{\mathcal{B}})$ is semi-free with a basis which we separate into three groups, as follows

$$\underbrace{\{E_{x,\mathcal{B}}\}_{x\in R_2} \cup \{sDf_{\mathcal{B}}\}_{f\in Q_2}}_{S_0^{\mathcal{B}}} \cup \underbrace{\{sDc_{x,\mathcal{A}}\}_{x\in R_1} \cup \{sDf_{\mathcal{A}}^{\vee}\}_{f\in Q_1}}_{S_1^{\mathcal{B}}} \cup \underbrace{\{sDc_{x,\mathcal{B}}\}_{x\in R_2} \cup \{sDf_{\mathcal{B}}^{\vee}\}_{f\in Q_2}}_{S_2^{\mathcal{B}}} \tag{23.4.20}$$

Similarly, the bimodule $\tilde{F}_!\mathrm{Res}(\Pi_{\mathcal{A}})$ is semi-free with basis which we separate into two groups

$$\underbrace{\{E_{x,\mathcal{A}}\}_{x\in R_1} \cup \{sDf_{\mathcal{A}}\}_{f\in Q_1}}_{S_0^{\mathcal{A}}} \cup \underbrace{\{\widetilde{sDc_{x,\mathcal{A}}}\}_{x\in R_1} \cup \{\widetilde{sDf_{\mathcal{A}}^{\vee}}\}_{f\in Q_1}}_{S_1^{\mathcal{A}}} \tag{23.4.21}$$

where we have decorated the basis elements $S_1^{\mathcal{A}}$ in $\tilde{F}_!\mathrm{Res}(\Pi_{\mathcal{A}})$ with $\widetilde{(-)}$ to avoid confusion with the generators with the same names in $\mathrm{Res}(\Pi_{\mathcal{B}})$.

The bimodule $\mathrm{Res}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ defined in (23.4.19) is therefore semi-free with basis given by

$$S_0^{\mathcal{B}} \cup S_1^{\mathcal{B}} \cup S_2^{\mathcal{B}} \cup sS_0^{\mathcal{A}} \cup sS_1^{\mathcal{A}}$$

where the basis elements of $\tilde{F}_!\mathrm{Res}(\Pi_{\mathcal{A}})$ are shifted by one.

Let M_i be the graded submodule generated by $S_i^{\mathcal{B}} \cup sS_i^{\mathcal{A}}$ for $i = 0, 1$, and let M_2 be the graded submodule generated by $S_2^{\mathcal{B}}$, then the differential in this cone preserves the filtration defined by $F_i = \oplus_{j\leq i} M_j$. Therefore, if we let $M = M_0 \oplus M_2$ and $M' = M_1$, then the pair (M, M') satisfies the conditions (1) and (2)

of Lemma 23.4.16. Thus, one has a (DG) bimodule (M_1, δ) over $\Pi_{\mathcal{B}}$. Moreover, we have the following

Lemma 23.4.22. *Let $H : M_1 \to M_1$ be the unique graded $\Pi_{\mathcal{B}}$-bilinear map such that*

$$\begin{aligned} sDc_{x,\mathcal{A}} &\mapsto \widetilde{ssDc_{x,\mathcal{A}}} & \qquad sDf_{\mathcal{A}}^{\vee} &\mapsto \widetilde{ssDf_{\mathcal{A}}^{\vee}} \\ \widetilde{ssDc_{x,\mathcal{A}}} &\mapsto 0 & \qquad \widetilde{ssDf_{\mathcal{A}}^{\vee}} &\mapsto 0 \end{aligned}$$

then $H\delta + \delta H = \mathrm{id}$.

Proof. Let $M_i^{\mathcal{B}}$ be the graded sub-bimodule of $\mathrm{Res}(\Pi_{\mathcal{B}})$ generated by the basis elements $S_i^{\mathcal{B}}$, then the same argument allows us to construct a bimodule $(M_1^{\mathcal{B}}, \delta^{\mathcal{B}})$. Similarly, the graded sub-bimodule $M_1^{\mathcal{A}}$ of $F_!(\mathrm{Res}(\Pi_{\mathcal{A}}))$ generated by the subset $S_1^{\mathcal{A}}$, also forms a bimodule $(M_1^{\mathcal{A}}, \delta^{\mathcal{A}})$ over $\Pi_{\mathcal{B}}$.

One can then show that the bimodule (M_1, δ) appearing in this lemma is the cone over the isomorphism $\gamma_F : (M_1^{\mathcal{A}}, \delta^{\mathcal{A}}) \to (M_1^{\mathcal{B}}, \delta^{\mathcal{B}})$. The lemma then follows. $\square$

Thus, the pair $(M, M') = (M_0 \oplus M_2\, ,\, M_1)$ satisfies all three conditions of Lemma 23.4.16. An application of Lemma 23.4.16 allows us to give a quasi-isomorphism

$$p : \mathrm{Res}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}}) = \mathrm{cone}\left[\tilde{F}_!\mathrm{Res}(\Pi_{\mathcal{A}}) \xrightarrow{\gamma_{\tilde{F}}} \mathrm{Res}(\Pi_{\mathcal{B}})\right] \xrightarrow{\sim} \overline{\mathrm{Res}}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}}) \tag{23.4.23}$$

where $\overline{\mathrm{Res}}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ is the reduction obtained by using Lemma 23.4.16 and 23.4.22 to 'kill' the generators $S_1^{\mathcal{B}} \cup sS_1^{\mathcal{A}}$ in $\mathrm{Res}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$.

Thus $\overline{\mathrm{Res}}(\Pi_{\mathcal{B}}, \Pi_{\mathcal{A}})$ has a basis set consisting of

$$\{sDc_{x,\mathcal{B}}\}_{x\in R_2} \cup \{sDf_{\mathcal{B}}^{\vee}\}_{f\in Q_2} \cup \{s^2Df_{\mathcal{A}}\}_{f\in Q_1} \cup \{sE_{x,\mathcal{A}}\}_{x\in R_1} \cup \{sDf_{\mathcal{B}}\}_{f\in Q_2} \cup \{E_{x,\mathcal{B}}\}_{x\in R_2} \tag{23.4.24}$$

Moreover, the map (23.4.23) is given by

$$
\begin{aligned}
p(E_{x,\mathcal{B}}) &= E_{x,\mathcal{B}} & p(sDf_{\mathcal{B}}) &= sDf_{\mathcal{B}} \\
p(sDf_{\mathcal{A}}^{\vee}) &= \beta(\widetilde{ssDf_{\mathcal{A}}^{\vee}}) & p(sDc_{x,\mathcal{A}}) &= \beta(\widetilde{ssDc_{x,\mathcal{A}}}) \\
p(sDf_{\mathcal{B}}^{\vee}) &= sDf_{\mathcal{B}}^{\vee} & p(sDc_{x,\mathcal{A}}) &= sDc_{x,\mathcal{A}} \\
p(sE_{x,\mathcal{A}}) &= sE_{x,\mathcal{A}} & p(s^2Df_{\mathcal{A}}) &= s^2Df_{\mathcal{A}} \\
p(\widetilde{ssDf_{\mathcal{A}}^{\vee}}) &= 0 & p(\widetilde{ssDc_{x,\mathcal{A}}}) &= 0
\end{aligned}
$$

Now, we show that the composition of (23.4.19) and (23.4.23) gives an isomorphism of bimodules. Indeed, the shift of the dual basis to (23.4.20) gives the following basis of $\mathrm{Res}(\Pi_{\mathcal{B}})^{\vee}[n]$

$$
\begin{gathered}
\{s^n(E_{x,\mathcal{B}})^{\vee}\}_{x\in R_2} \cup \{s^n(sDf_{\mathcal{B}})^{\vee}\}_{f\in Q_2} \cup \{s^n(sDf_{\mathcal{A}}^{\vee})^{\vee}\}_{f\in Q_1} \cup \{s^n(sDc_{x,\mathcal{A}})^{\vee}\}_{x\in R_1} \\
\cup \{s^n(sDf_{\mathcal{B}}^{\vee})^{\vee}\}_{f\in Q_2} \cup \{s^n(sDc_{x,\mathcal{B}})^{\vee}\}_{x\in R_2}
\end{gathered}
\tag{23.4.25}
$$

With respect to the basis (23.4.25) and (23.4.24), the composition

$$
\mathrm{Res}(\Pi_{\mathcal{B}})^{\vee}[n] \xrightarrow{\tilde{\xi}_{\mathcal{B}}s^{-n}} \mathrm{Res}(\Pi_{\mathcal{B}},\Pi_{\mathcal{A}}) \xrightarrow{p} \overline{\mathrm{Res}}(\Pi_{\mathcal{B}},\Pi_{\mathcal{A}})
$$

of (23.4.19) and (23.4.23) is then given by the matrix

$$
\begin{bmatrix}
\pm1 & 0 & 0 & 0 & 0 & 0 \\
0 & \pm1 & 0 & 0 & 0 & 0 \\
* & * & \pm1 & 0 & 0 & 0 \\
* & * & 0 & \pm1 & 0 & 0 \\
* & * & 0 & 0 & \pm1 & 0 \\
* & * & 0 & 0 & 0 & \pm1
\end{bmatrix}
$$

which is therefore an isomorphism of bimodules. This finishes the proof of Lemma 23.4.15, and hence Theorem 23.4.14. □

CHAPTER 24

BAUES-LEMAIRE CYLINDERS AND VARIATIONS

In this section, we construct canonical cylinder objects for Sullivan DG categories. This extends and generalizes the construction in [13] in the case of DG algebras. We also give a "directed" version of this construction. These constructions are applied to give descriptions of Calabi-Yau completions of DG categories that are themselves Calabi-Yau.

24.1 Directed Baues-Lemaire cylinder

For any bimodule $M \in \mathcal{C}(\mathcal{A}^{\mathrm{e}})$ over a DG category $\mathcal{A}$, we will now define a bimodule M_{21} over the DG category $\mathcal{A} \amalg \mathcal{A}$ formed by the disjoint union of $\mathcal{A}$ with itself. For convenience of notation, we will denote this disjoint union as $\mathcal{A} \amalg \mathcal{A}'$ to distinguish the second (identical) copy with the first.

Then the enveloping DG category $(\mathcal{A} \amalg \mathcal{A}')^{\mathrm{e}}$ can be written as the disjoint union

$$(\mathcal{A} \amalg \mathcal{A}')^{\mathrm{e}} = (\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}) \amalg (\mathcal{A} \otimes \mathcal{A}'^{\,\mathrm{op}}) \amalg (\mathcal{A}' \otimes \mathcal{A}^{\mathrm{op}}) \amalg (\mathcal{A}' \otimes \mathcal{A}'^{\,\mathrm{op}})$$

The bimodule M_{21} is then defined as the DG functor

$$M_{21} : (\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}) \amalg (\mathcal{A} \otimes \mathcal{A}'^{\,\mathrm{op}}) \amalg (\mathcal{A}' \otimes \mathcal{A}^{\mathrm{op}}) \amalg (\mathcal{A}' \otimes \mathcal{A}'^{\,\mathrm{op}}) \xrightarrow{(0,0,M,0)} \mathcal{C}(k) \quad (24.1.1)$$

Explicitly, for each object $x, y \in \mathrm{Ob}(\mathcal{A})$, denote by x', y' the corresponding object in the identical copy $\mathcal{A}'$, then we have

$$M_{21}(x,y) = M_{21}(x,y') = M_{21}(x',y') = 0 \qquad M_{21}(x',y) = M(x,y)$$

Thus, if M is semi-free [resp. Sullivan] over $\mathcal{A}$, then M_{21} is semi-free [resp. Sullivan] over $\mathcal{A} \amalg \mathcal{A}'$.

Now, we consider the tensor category $T_{\mathcal{A}\amalg\mathcal{A}'}(M_{21})$, which we call the *directed tensor category* of M over $\mathcal{A}$. Notice that, since M_{21} points from $\mathcal{A}'$ to $\mathcal{A}$ in the directed tensor category, there is no compositions involving two elements from M. Therefore, one has the following simple description of the directed tensor category

$$\begin{aligned} T_{\mathcal{A}\amalg\mathcal{A}'}(M_{21})(x,y) &= \mathcal{A}(x,y) & T_{\mathcal{A}\amalg\mathcal{A}'}(M_{21})(x',y') &= \mathcal{A}'(x',y') = \mathcal{A}(x,y) \\ T_{\mathcal{A}\amalg\mathcal{A}'}(M_{21})(x',y) &= M(x,y) & T_{\mathcal{A}\amalg\mathcal{A}'}(M_{21})(x,y') &= 0 \end{aligned} \tag{24.1.2}$$

One could write this as

$$T_{\mathcal{A}\amalg\mathcal{A}'}(M_{21}) = \begin{bmatrix} \mathcal{A} & M \\ 0 & \mathcal{A}' \end{bmatrix}$$

Therefore, if $f : M \overset{\sim}{\to} N$ is a quasi-isomorphism of bimodules, then the explicit description (24.1.2) shows that the induced map

$$T_{\mathcal{A}\amalg\mathcal{A}'}(M_{21}) \overset{\sim}{\to} T_{\mathcal{A}\amalg\mathcal{A}'}(N_{21}) \tag{24.1.3}$$

is a quasi-equivalence of DG categories.

In particular, consider the standard bimodule resolution $\mathrm{Res}(\mathcal{A}) \overset{\sim}{\to} \mathcal{A}$, then the induced map

$$T_{\mathcal{A}\amalg\mathcal{A}'}(\,\mathrm{Res}(\mathcal{A})_{21}\,) \overset{\sim}{\to} T_{\mathcal{A}\amalg\mathcal{A}'}(\,\mathcal{A}_{21}\,) \tag{24.1.4}$$

on the directed tensor categories is a quasi-equivalence. Thus, if $\mathcal{A}$ is Sullivan, the (24.1.4) gives a canonical Sullivan resolution of the DG category $T_{\mathcal{A}\amalg\mathcal{A}'}(\,\mathcal{A}_{21}\,)$.

Definition 24.1.5. Suppose $\mathcal{A}$ is a semi-free DG category, then the DG category

$$\overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A}) := T_{\mathcal{A} \amalg \mathcal{A}'}(\, \mathrm{Res}(\mathcal{A})_{21} \,)$$

is called the *directed Baues-Lemaire cylinder* of $\mathcal{A}$.

One can slightly modify the construction of the quasi-equivalence (24.1.4) by shifting the quasi-isomorphism $\mathrm{Res}(\mathcal{A}) \xrightarrow{\sim} \mathcal{A}$ of bimodules by an integer p. Then (24.1.3) again gives a quasi-equivalence

$$T_{\mathcal{A} \amalg \mathcal{A}'}(\, \mathrm{Res}(\mathcal{A})_{21}[p] \,) \xrightarrow{\sim} T_{\mathcal{A} \amalg \mathcal{A}'}(\, \mathcal{A}_{21}[p] \,) \tag{24.1.6}$$

Definition 24.1.7. Suppose $\mathcal{A}$ is a semi-free DG category, then the DG category

$$\overrightarrow{\mathrm{Cyl}}^{[p]}_{\mathrm{BL}}(\mathcal{A}) := T_{\mathcal{A} \amalg \mathcal{A}'}(\, \mathrm{Res}(\mathcal{A})_{21}[p] \,)$$

is called the *p-shifted directed Baues-Lemaire cylinder* of $\mathcal{A}$.

We now give an alternative description of both sides in the quasi-equivalence (24.1.6). In particular, when $p = 0$, this gives a description of (24.1.4).

Let $\vec{I}^{[p]}$ be the DG category with two objects 0 and 1, such that

$$\vec{I}^{[p]}(0,0) = \vec{I}^{[p]}(1,1) = k \qquad\qquad \vec{I}^{[p]}(0,1) = k[p]$$

Thus, $\vec{I}^{[p]}$ is semi-free over the graded quiver $[\, 0 \xrightarrow{v} 1 \,]$, where $|v| = p$. We call $\vec{I}^{[p]}$ the *p-shifted directed interval*. When $p = 0$, we simply call it the *directed interval*, and denote it as $\vec{I}$

Then, from the explicit description (24.1.2) of the directed tensor category, it is easy to see that

$$T_{\mathcal{A} \amalg \mathcal{A}'}(\mathcal{A}[p]) \cong \mathcal{A} \otimes \vec{I}^{[p]} \tag{24.1.8}$$

Thus, if $\mathcal{A}$ is semi-free, then (24.1.4) and (24.1.8) can be combined to show that the directed Baues-Lemaire cylinder (Definition 24.1.5) gives a canonical semi-free resolution

$$\overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A}) \xrightarrow{\sim} A \otimes \vec{I} \tag{24.1.9}$$

of the tensor product of $\mathcal{A}$ with the directed interval $\vec{I}$. This justifies why we called $\overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A})$ the 'directed cylinder' in Definition 24.1.5. The same holds true for the p-shifted case.

To explain why this directed cylinder is called the Baues-Lemaire directed cylinder, we will now give a more explicit description of it when $\mathcal{A}$ is semi-free over a graded quiver (R, Q). In this case, the formulas for the differentials will be completely parallel to those for the Baues-Lemaire cylinder in [13]. The exact relation will be explained in the next subsection.

We will work directly with the p-shifted case, since this generality requires no extra effort. Thus, suppose $\mathcal{A} = T_R(Q)$ is semifree over a graded quiver (R, Q). Then the standard resolution $\mathrm{Res}(\mathcal{A})$ is semi-free over the basis (22.5.7). Therefore, the bimodule M_{21} over $\mathcal{A} \amalg \mathcal{A}'$ is also semi-free with the same basis set, now considered as elements in $M_{21}(x', y) = M(x, y),\ x, y \in \mathrm{Ob}(\mathcal{A})$. If we rename the basis elements of $\mathrm{Res}(\mathcal{A})[p]$ as $[f] := s^p(sDf)$ and $[E_x] := s^p E_x$, then the directed tensor category $T_{\mathcal{A} \amalg \mathcal{A}'}(\mathrm{Res}(\mathcal{A})[p])$ is semi-free, with a description

$$T_{\mathcal{A} \amalg \mathcal{A}'}(\mathrm{Res}(\mathcal{A})[p]) = T_{R \amalg R'}\big(\{f\}_{f \in Q} \cup \{f'\}_{f \in Q} \cup \{[f]\}_{f \in Q} \cup \{[E_x]\}_{x \in R}\big) \tag{24.1.10}$$

In this description, each arrow $f : x \to y$ in the quiver Q determines three generators $f, f', [f]$. These generators point respectively in the directions $f : x \to y,\ \ f' : x' \to y'$, and $[f] : x' \to y$. The generators f and f' have the same degree as the corresponding arrow f in Q. The generator $[f]$ has degree

$\deg([f]) = \deg(f) + p + 1$. For each $x \in R$, there is also a generator $[E_x]$ of degree p, pointing from x' to x.

Since there is, by definition, an injective map $\mathcal{A} \amalg \mathcal{A}' \to T_{\mathcal{A} \amalg \mathcal{A}'}(\mathrm{Res}(\mathcal{A})[p])$, the generators f and f' have differentials completely the same as the corresponding generators in $\mathcal{A}$. The extra generators $[E_x]$ have zero differentials.

To determine the differentials of the extra generators $[f]$, suppose that, in the DG category $\mathcal{A}$, we have

$$d(f) = \sum_{I=(i_1,\ldots,i_r)} a_I \cdot g_{i_1} \ldots g_{i_r}$$

where $a_I \in k$ are coefficients, and I runs over all composable sequences $g_{i_1} \ldots g_{i_r}$ of arrows in the quiver Q. Then, the differential of the extra generator $[f]$ is given by

$$\begin{aligned} d([f]) = (-1)^p \; &\big((-1)^{p|f|} f \cdot [E_x] - [E_y] \cdot f \big) \\ &+ (-1)^{p+1} \sum_{I=(i_1,\ldots,i_r)} \sum_{j=1}^{r} (-1)^{(p+1)|g_{i_1} \ldots g_{i_{j-1}}|} a_I \cdot g_{i_1} \ldots g_{i_{j-1}} \cdot [g_{i_j}] \cdot g'_{i_{j+1}} \ldots g'_{i_r} \end{aligned} \tag{24.1.11}$$

When $p = 0$, this formula is the same as the formula for the differentials of the extra generators in the Baues-Lemaire cylinder [13]. We explain the exact relation in the next subsection.

24.2 Relation with the original Baues-Lemaire cylinder

Recall that the directed interval $\vec{I}$ is defined to be the unique DG category semi-free over the quiver $[\,0 \xrightarrow{v} 1\,]$. This is in contrast to the (undirected) interval I, which is almost semi-free over the same quiver, except that v is inverted. *i.e.*, we have $I = \vec{I}[v^{-1}]$.

The interval I is, in fact, a cylinder object for the unit $*$ of the monoidal product $\otimes$ on $\mathtt{dgCat}_k$, in the sense that there is a factorization of the folding map $\nabla = (\mathrm{id}, \mathrm{id}) : * \amalg * \to *$ into

$$* \amalg * \hookrightarrow I \xrightarrow{\sim} *$$

Notice that tensor product of DG categories preserves quasi-equivalences provided that at least one side of each of the tensor products is k-flat. Therefore, tensoring the above factorization with any DG category $\mathcal{A}$ gives a factorization of the folding map $\nabla = (\mathrm{id}, \mathrm{id}) : \mathcal{A} \amalg \mathcal{A} \to \mathcal{A}$ into

$$\mathcal{A} \amalg \mathcal{A} \to \mathcal{A} \otimes I \xrightarrow{\sim} \mathcal{A} \tag{24.2.1}$$

The map $\mathcal{A} \amalg \mathcal{A} \to \mathcal{A}$ in (24.2.1) is not a cofibration. In order to obtain a cylinder object for $\mathcal{A}$, one should further factorize this map into a cofibration followed by a weak equivalence.

In the last subsection, we have achieved something similar. Namely, if we consider the directed interval $\vec{I}$ instead of the undirected interval I, then the canonical map $\mathcal{A} \amalg \mathcal{A} \to \mathcal{A} \otimes \vec{I}$ has a canonical factorization

$$\mathcal{A} \amalg \mathcal{A} \to \overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A}) \xrightarrow{\sim} \mathcal{A} \otimes \vec{I}$$

where the map $\overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A}) \xrightarrow{\sim} \mathcal{A} \otimes \vec{I}$ is given by (24.1.9). Moreover, if the DG category $\mathcal{A}$ is semi-free, then the map $\mathcal{A} \amalg \mathcal{A} \to \overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A})$ is a semi-free extension, with an explicit description (24.1.10).

We wish to have a parallel version of this construction when the directed interval $\vec{I}$ is replaced by the undirected interval I. Since I is a localization of $\vec{I}$

by inverting the arrow v, it is natural to expect that the directed Baues-Lemaire cylinder $T_{\mathcal{A}\amalg\mathcal{A}'}(\mathrm{Res}(\mathcal{A}))$ should be replaced by a suitable localization of it. This is indeed the case.

Definition 24.2.2. The *Baues-Lemaire cylinder* of a semi-free DG category $\mathcal{A} = T_R(Q)$ is the localization of the directed Baues-Lemaire cylinder $\overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A})$ at the generators $[E_x]$ of (24.1.10).

$$\mathrm{Cyl}_{\mathrm{BL}}(\mathcal{A}) \;:=\; \overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A})\big[\,[E_x]^{-1}\,\big]_{x\in R}$$

This DG category has the same description as in (24.1.10) for $p = 0$, except that the generators $[E_x]$ are now invertible. *i.e.*, we have

$$\mathrm{Cyl}_{\mathrm{BL}}(\mathcal{A}) \;=\; T_{R\amalg R'}\big(\,\{f\}_{f\in Q}\,\cup\,\{f'\}_{f\in Q}\,\cup\,\{[f]\}_{f\in Q}\,\cup\,\{[E_x]^{\pm}\}_{x\in R}\,\big) \qquad (24.2.3)$$

with the same formulas for the differentials of the generators (see (24.1.11)).

The canonical map $\overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A}) \;\xrightarrow{\sim}\; \mathcal{A}\otimes\vec{I}$ for the directed cylinder induces a map

$$\mathrm{Cyl}_{\mathrm{BL}}(\mathcal{A}) \;\to\; \mathcal{A}\otimes I \qquad (24.2.4)$$

by passing to the localizations on both sides.

Theorem 24.2.5. *The map* (24.2.4) *is a quasi-equivalence.*

By (24.2.1) and the discussion that follows, the Baues-Lemaire cylinder is then a cylinder object for $\mathcal{A}$ in the case when $\mathcal{A}$ is Sullivan. This justifies the name Baues-Lemaire cylinder of $\mathrm{Cyl}_{\mathrm{BL}}(\mathcal{A})$.

To prove Theorem 24.2.5, we relate the notion of Baues-Lemaire cylinder in Definition 24.2.2 with the original Baues-Lemaire cylinder in [13]. To do this, observe that, in the Baues-Lemaire cylinder (24.2.3) of $\mathcal{A} = T_R(Q)$, the generators

$[E_x]$ are invertible generators joining distinct objects $x' \in R'$ and $x \in R$. Therefore, contracting this arrow does not affect the quasi-equivalence type. This leads to the following

Definition 24.2.6. The *fixed object-set Baues-Lemaire cylinder* of a semi-free DG category $\mathcal{A} = T_R(Q)$, denoted as $\mathrm{Cyl}^R_{\mathrm{BL}}(\mathcal{A})$, is the result of contracting the invertible maps $[E_x]$ in the Baues-Lemaire cylinder $\mathrm{Cyl}_{\mathrm{BL}}(\mathcal{A})$ to the identity.

Therefore, the fixed object-set Baues-Lemaire cylinder has an explicit description

$$\mathrm{Cyl}^R_{\mathrm{BL}}(\mathcal{A}) \;=\; T_R\big(\{f\}_{f\in Q} \,\cup\, \{f'\}_{f\in Q} \,\cup\, \{[f]\}_{f\in Q}\big) \tag{24.2.7}$$

where the extra generators have differentials given by (24.1.11) for $p = 0$.

In particular, when $R = \{*\}$ is a singleton, the DG category $\mathcal{A}$ is simply a semi-free DG algebra. Then the fixed object-set Baues-Lemaire cylinder of $\mathcal{A}$ coincides precisely with the original Baues-Lemaire cylinder in [13].

Now, consider the commutative diagram of canonical maps

$$\begin{tikzcd}
\mathrm{Cyl}_{\mathrm{BL}}(\mathcal{A}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Cyl}^R_{\mathrm{BL}}(\mathcal{A}) \arrow[d] \\
\mathcal{A} \otimes I \arrow[r, "\sim"] & \mathcal{A}
\end{tikzcd}$$

where the maps indicated with the arrow $\xrightarrow{\sim}$ are known to be quasi-equivalences by the above discussion. Thus, to show Theorem 24.2.5, one suffices to show that the canonical map $\mathrm{Cyl}^R_{\mathrm{BL}}(\mathcal{A}) \to \mathcal{A}$ is a quasi-equivalence.

In the case when R is a singleton, this statement was shown in [13] (see also [5]). In fact, the proof in [5] carries over mutatis mutandis to the general case of an arbitrary object set R. This completes the proof of Theorem 24.2.5.

Remark 24.2.8. The Baues-Lemaire cylinder (Definition 24.2.2) can be used to construct mapping cylinders that factorizes any given DG functor $F : \mathcal{A} \to \mathcal{B}$ between Sullivan DG categories into a composition $\mathcal{A} \hookrightarrow \mathrm{Cyl}_{\mathrm{BL}}(F) \to \mathcal{B}$ where $\mathcal{A} \hookrightarrow \mathrm{Cyl}_{\mathrm{BL}}(F)$ is a semi-free extension. This mapping cylinder is defined as a pushout in the obvious way, in analogy with the topological definition. Using Theorem 24.2.5, together with Propositions A.0.4 and A.0.8, one can show that the canonical map $\mathrm{Cyl}_{\mathrm{BL}}(F) \to \mathcal{B}$ is a quasi-equivalence. Thus, one can replace any DG functor between Sullivan DG categories by a semi-free extension that is quasi-equivalent to the original DG functor.

24.3 Application to Calabi-Yau completions

Suppose that a finitely Sullivan DG category $\mathcal{A}$ has an m-Calabi-Yau structure $[\xi_{\mathcal{A}}] \in \mathrm{HC}^{-}_{m}(\mathcal{A})$, we consider the $(p+m+1)$-Calabi-Yau completion of $\mathcal{A}$.

Choose any representative $\xi^{(0)}_{\mathcal{A}} \in \mathbb{X}(\mathcal{A})$ in the double X-complex of the underlying weak m-Calabi-Yau structure $[\xi^{(0)}_{\mathcal{A}}] = h([\xi_{\mathcal{A}}])$. Then $\xi_{\mathcal{A}}$ determines a degree zero closed map

$$\hat{\xi}^{(0)}_{\mathcal{A}} s^{-m} \,:\, \mathrm{Res}(\mathcal{A})^{\vee}[m] \,\to\, \mathrm{Res}(\mathcal{A})$$

By definition of a Calabi-Yau structure, this map is a quasi-isomorphism. Therefore, the $(p+m+1)$-Calabi-Yau completion of $\mathcal{A}$ is quasi-equivalent to another tensor category

$$\Pi_{p+m+1}(\mathcal{A}) \,=\, T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})^{\vee}[p+m]\,) \xrightarrow{T_{\mathcal{A}}(\hat{\xi}^{(0)}_{\mathcal{A}} s^{-m})} T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})[p]) \tag{24.3.1}$$

Notice that the tensor category $T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})[p]\,)$ can be written as a pushout

$$T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})[p]\,) \;=\; \mathrm{colim}\,\big[\,\mathcal{A} \xleftarrow{\nabla} \mathcal{A} \amalg \mathcal{A}' \xrightarrow{(i,i')} T_{\mathcal{A}\amalg\mathcal{A}'}\big(\,\mathrm{Res}(\mathcal{A})_{21}[p]\,\big)\,\big]$$

The last term in this pushout is the p-shifted directed Baues-Lemaire cylinder of $\mathcal{A}$ (see Definition 24.1.7). By the discussion in Section 24.1, this directed cylinder is quasi-equivalent to the tensor product $\mathcal{A} \otimes \vec{I}^{[p]}$ of $\mathcal{A}$ with the shifted interval (see (24.1.6), (24.1.8)). Thus, we have

$$\begin{aligned} T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})[p]\,) \;&=\; \mathrm{colim}\,\big[\,\mathcal{A} \xleftarrow{\nabla} \mathcal{A} \amalg \mathcal{A}' \xrightarrow{(i,i')} T_{\mathcal{A}\amalg\mathcal{A}'}\big(\,\mathrm{Res}(\mathcal{A})_{21}[p]\,\big)\,\big] \\ &=\; \mathrm{hocolim}\,\big[\,\mathcal{A} \xleftarrow{\nabla} \mathcal{A} \amalg \mathcal{A}' \xrightarrow{(i,i')} \mathcal{A} \otimes \vec{I}^{[p]}\,\big] \end{aligned} \tag{24.3.2}$$

Now, the last diagram is simply given by tensoring $\mathcal{A}$ with the diagram

$$\big[\,* \xleftarrow{\nabla} * \amalg * \xrightarrow{(i,i')} \vec{I}^{[p]}\,\big]$$

Thus, in order to compute the homotopy colimit in (24.3.2), we quote the following result in [155, Corollary 6.7].

Proposition 24.3.3. *For any fixed $\mathcal{A} \in \mathtt{dgCat}_k$, the derived tensor product*

$$\mathcal{A} \otimes^{\boldsymbol{L}} (-) \;:\; Ho(\mathtt{dgCat}_k) \;\to\; Ho(\mathtt{dgCat}_k)$$

preserves homotopy colimits.

Applying this proposition to (24.3.2), we have

$$\begin{aligned} T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})[p]\,) \;&=\; \mathcal{A} \otimes \mathrm{hocolim}\,\big[\,* \xleftarrow{\nabla} * \amalg * \xrightarrow{(i,i')} \vec{I}^{[p]}\,\big] \\ &=\; \mathcal{A} \otimes k\langle z\rangle \end{aligned} \tag{24.3.4}$$

where z is a degree p element, and the graded associative algebra $k\langle z\rangle$ is regarded as a DG category with one object.

In other words, we have

Proposition 24.3.5. *The natural map*

$$\begin{array}{ccc} T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})[p]\,) & \longrightarrow & \mathcal{A}\otimes k\langle z\rangle \\ s^p E_x & \longmapsto & \mathrm{id}_x \otimes z \\ s^{p+1} Df & \longmapsto & 0 \end{array}$$

is a quasi-equivalence.

By the above discussion (see (24.3.1), (24.3.4)), the $(p+m+1)$-Calabi-Yau completion of $\mathcal{A}$ is quasi-equivalent to the tensor product $\mathcal{A} \otimes k\langle z\rangle$.

$$\Pi_{p+m+1}(\mathcal{A}) \;=\; T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})^{\vee}[p+m]) \;\xrightarrow{\sim}\; T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})[p]) \;\xrightarrow{\sim}\; \mathcal{A} \otimes k\langle z\rangle \qquad (24.3.6)$$

Under this quasi-equivalence, the canonical $(p+m+1)$-Calabi-Yau structure on the $(p+m+1)$-Calabi-Yau completion of $\mathcal{A}$ induces a $(p+m+1)$-Calabi-Yau structure on the tensor product $\mathcal{A} \otimes k\langle z\rangle$. This structure has a simple description.

To give this description, we first construct a canonical $(p+1)$-Calabi-Yau structure on the algebra $k\langle z\rangle$. Consider the reduced bar complex

$$\overline{C}^{\mathrm{bar}}(\,k\langle z\rangle\,) = \bigoplus_{n\geq 0} k\langle z\rangle \,\otimes\, \left(\overline{k\langle z\rangle}[1]\right)^{\otimes n}$$

The element z can be regarded as an element (z) in the $n=0$ component of the reduced bar complex. This element is closed in the reduced bar complex and hence defines a class $[(z)] \in \mathrm{HH}_p(k\langle z\rangle)$. The image under $B : \mathrm{HH}_p(k\langle z\rangle) \to \mathrm{HC}^-_{p+1}(k\langle z\rangle)$ is then represented by the element

$$B(z) = 1 \otimes sz \;\in\; k\langle z\rangle \otimes \overline{k\langle z\rangle}[1] \;\subset\; \overline{C}^{\mathrm{bar}}(k\langle z\rangle) \;\subset\; \overline{C}^{\mathrm{bar}}(k\langle z\rangle)[[u]] \qquad (24.3.7)$$

Definition 24.3.8. The *fundamental class* of the graded associative algebra $k\langle z\rangle$ is the negative cyclic class of $[B(z)] \in \mathrm{HC}^-_{p+1}(k\langle z\rangle)$ represented by the element $B(z)$ in (24.3.7).

It is easy to see that the fundamental class of $k\langle z\rangle$ is a $(p+1)$-Calabi-Yau structure on $k\langle z\rangle$. In fact, the fundamental class on $k\langle z\rangle$ can be used to describe the canonical $(p+m+1)$-Calabi-Yau structure on $\mathcal{A}\otimes k\langle z\rangle$ induced by the quasi-equivalence (24.3.6).

Proposition 24.3.9. *Under the quasi-equivalence* (24.3.6)*, the canonical* $(p+m+1)$*-Calabi-Yau structure on the* $(p+m+1)$*-Calabi-Yau completion* $\Pi_{p+m+1}(\mathcal{A})$ *corresponds to the class*

$$\mathsf{Sh}^{-}(\,[\xi_{\mathcal{A}}]\otimes[B(z)]\,)\;\in\;\mathrm{HC}^{-}_{p+m+1}(\mathcal{A}\otimes k\langle z\rangle)$$

given by the cyclic shuffle product (22.4.13) *of the given* m*-Calabi-Yau structure* $[\xi_{\mathcal{A}}]\in\mathrm{HC}^{-}_{m}(\mathcal{A})$ *on* $\mathcal{A}$ *with the fundamental class* $[B(z)]\in\mathrm{HC}^{-}_{p+1}(k\langle z\rangle)$ *on* $k\langle z\rangle$*.*

Proof. We will work with the reduced bar complexes as models for Hochschild complexes. Then the map (23.3.5) is replaced by the map

$$j'_{\mathcal{A}}\;:\;\overline{C}^{\mathrm{bar}}(\,\mathcal{A}\,;\,\mathrm{Res}(\mathcal{A})^{\vee}[m+p]\,)\;\to\;\overline{C}^{\mathrm{bar}}(\,T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})^{\vee}[m+p]\,)\,)$$

induced by the inclusion $\mathrm{Res}(\mathcal{A})^{\vee}[m+p]\;\hookrightarrow\;T_{\mathcal{A}}(\mathrm{Res}(\mathcal{A})^{\vee}[m+p])$.

Similarly, the closed map of $\mathcal{A}$-bimodules

$$\mathcal{A}[p]\;\to\;A\otimes k\langle z\rangle,\qquad\qquad f\;\mapsto\;f\otimes z$$

induces a closed map of reduced bar complexes

$$j''_{\mathcal{A}}\;:\;\overline{C}^{\mathrm{bar}}(\mathcal{A};\mathcal{A}[p])\;\to\;\overline{C}^{\mathrm{bar}}(A\otimes k\langle z\rangle)\tag{24.3.10}$$

These two maps $j'_{\mathcal{A}}$ and $j''_{\mathcal{A}}$ induce maps at the homology level, which fit together into a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{\bullet}(\,\mathcal{A}\,;\,\mathcal{A}^{!}[m+p]\,) \arrow[r, "s^{p}[\hat{\xi}^{(0)}_{\mathcal{A}}]s^{-m-p}", "\cong"'] \arrow[d, "j'_{\mathcal{A}}"'] & \mathrm{HH}_{\bullet}(\,\mathcal{A}\,;\,\mathcal{A}[p]\,) \arrow[d, "j''_{\mathcal{A}}"] \\
\mathrm{HH}_{\bullet}(\,T_{\mathcal{A}}(\mathrm{Res}(\mathcal{A})^{\vee}[m+p])\,) \arrow[r, "(24.3.6)", "\cong"'] & \mathrm{HH}_{\bullet}(\mathcal{A}\otimes k\langle z\rangle)
\end{tikzcd}\tag{24.3.11}$$

where the horizontal map in the first row is the map induced by the underlying weak Calabi-Yau structure $[\xi_{\mathcal{A}}^{(0)}] \in \mathrm{HH}_m(\mathcal{A})$ of $\mathcal{A}$. The image of the shifted Casimir element $s^{m+p}[\theta_{\mathcal{A}}] \in \mathrm{HH}_{m+p}(\,\mathcal{A}\,;\,\mathcal{A}^![m+p]\,)$ under this map is therefore simply given by the p-shift $s^p[\xi_{\mathcal{A}}^{(0)}] \in \mathrm{HH}_{\bullet}(\,\mathcal{A}\,;\,\mathcal{A}[p]\,)$ of this weak Calabi-Yau structure. This shows that the induced $(m+p+1)$-Calabi-Yau structure on $\mathcal{A} \otimes k\langle z \rangle$ is given by

$$B\,(\,j''_{\mathcal{A}}(\,s^p\,[\xi_{\mathcal{A}}^{(0)}]\,)\,) \,\in\, \mathrm{HC}^-_{p+m+1}(\mathcal{A} \otimes k\langle z \rangle) \tag{24.3.12}$$

To calculate the element (24.3.12), we suppose that the class $s^p[\xi_{\mathcal{A}}^{(0)}]$ is represented by a sum of the following form in the reduced bar complex

$$s^p \xi_{\mathcal{A}}^{(0)} \,=\, \sum (s^p f_0) \otimes \overline{f_1} \otimes \ldots \otimes \overline{f_r} \qquad \in\, \overline{C}^{\mathrm{bar}}(\,\mathcal{A}\,;\,\mathcal{A}[p]\,)$$

then the image of this element under (24.3.10) is clearly given by

$$j''_{\mathcal{A}}(\,s^p\,\xi_{\mathcal{A}}^{(0)}) \,=\, \sum (f_0 \otimes z) \otimes \overline{(f_1 \otimes 1)} \otimes \ldots \otimes \overline{(f_r \otimes 1)} \qquad \in\, \overline{C}^{\mathrm{bar}}(\,\mathcal{A} \otimes k\langle z \rangle\,)$$

In other words, the element $j''_{\mathcal{A}}(\,s^p\,\xi_{\mathcal{A}}^{(0)}) \,\in\, \overline{C}^{\mathrm{bar}}(\,\mathcal{A} \otimes k\langle z \rangle\,)$ is given by the shuffle product

$$j''_{\mathcal{A}}(\,s^p\,\xi_{\mathcal{A}}^{(0)}) \,=\, \mathrm{sh}(\,\xi_{\mathcal{A}}^{(0)} \,\otimes\, (z)\,) \tag{24.3.13}$$

between the elements $\xi_{\mathcal{A}}^{(0)} \in \overline{C}^{\mathrm{bar}}(\,\mathcal{A}\,)$ and $(z) \in \overline{C}^{\mathrm{bar}}(\,k\langle z \rangle\,)$, both considered to sit inside the reduced bar complex $\overline{C}^{\mathrm{bar}}(\,\mathcal{A} \otimes k\langle z \rangle\,)$ of $\mathcal{A} \otimes k\langle z \rangle$.

Thus, it suffices to show that the element $B(\,\mathrm{sh}(\,\xi_{\mathcal{A}}^{(0)} \otimes (z)\,)\,) \,\in\, \overline{C}^{\mathrm{bar}}(\,\mathcal{A} \otimes k\langle z \rangle\,)$ and the power series $\mathsf{Sh}^-(\,\xi_{\mathcal{A}} \,\otimes\, B(z)\,) \,\in\, \overline{C}^{\mathrm{bar}}(\,\mathcal{A} \otimes k\langle z \rangle\,)[[u]]$ represent the same class in the negative cyclic homology $\mathrm{HC}^-_{m+p+1}(\mathcal{A} \otimes k\langle z \rangle)$.

To this end, we compute the two terms

$$uB(\,\mathrm{sh}(\,\xi_{\mathcal{A}}^{(0)} \,\otimes\, (z)\,)\,) = (b + uB)(\,\mathrm{sh}(\,\xi_{\mathcal{A}}^{(0)} \,\otimes\, (z)\,)\,)$$

and

$$
\begin{aligned}
u\mathsf{Sh}^{-}(\,\xi_{\mathcal{A}} \otimes B(z)\,) &= \mathsf{Sh}^{-}(\,\xi_{\mathcal{A}} \otimes uB(z)\,) \\
&= \mathsf{Sh}^{-}(\,\xi_{\mathcal{A}} \otimes (b+uB)(z)\,) \\
&= \mathsf{Sh}^{-}(((b+uB)\otimes \mathrm{id} + \mathrm{id}\otimes (b+uB))(\,\xi_{\mathcal{A}} \otimes (z)\,)) \\
&= (b+uB)(\,\mathsf{Sh}^{-}(\,\xi_{\mathcal{A}} \otimes (z)\,)\,)
\end{aligned}
$$

Notice that the difference

$$
\mathrm{sh}(\,\xi_{\mathcal{A}}^{(0)} \otimes (z)\,) - \mathsf{Sh}^{-}(\,\xi_{\mathcal{A}} \otimes (z)\,)
$$

have zero constant term, and hence is divisible by u. Therefore, we have

$$
B(\,\mathrm{sh}(\,\xi_{\mathcal{A}}^{(0)} \otimes (z)\,)\,) - \mathsf{Sh}^{-}(\,\xi_{\mathcal{A}} \otimes B(z)\,) = (b+uB)\left(\frac{\mathrm{sh}(\,\xi_{\mathcal{A}}^{(0)} \otimes (z)\,) - \mathsf{Sh}^{-}(\,\xi_{\mathcal{A}} \otimes (z)\,)}{u}\right)
$$

This completes the proof.

□

24.4 Localized relative Calabi-Yau completion

In the last subsection, we have shown that, if $\mathcal{A}$ has an m-Calabi-Yau structure, then its $(m+p+1)$-Calabi-Yau completion can be formed by collapsing the two copies of $\mathcal{A}$ in the p-shifted directed Baues-Lemaire cylinder (see (24.3.2)).

When $p = 0$, we have seen in Section 24.2 that inverting the generating arrows $\{\,[E_x]\,\}_{x\in R}$ in the directed Baues-Lemaire cylinder gives the (undirected) Baues-Lemaire cylinder. Thus, replacing the directed Baues-Lemaire cylinder in (24.3.2) with the undirected one, we will get a description of a localization of the Calabi-Yau completion of $\mathcal{A}$. This will be applied in the next section to relate Calabi-Yau completions with topology.

We start with the following

Definition 24.4.1. Let $\mathcal{A} = T_R(Q)$ be finitely Sullivan with a given m-Calabi-Yau structure $[\xi_{\mathcal{A}}] \in \mathrm{HC}^-_m(\mathcal{A})$, we define the *localized* $(m+1)$-*Calabi-Yau completion* of $(\mathcal{A}, [\xi_{\mathcal{A}}])$ to be the localized tensor category

$$\Pi^{\mathrm{loc}}_{m+1}(\mathcal{A}) \,:=\, T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})\,)[\,E_x^{-1}\,]_{x\in R}$$

By (24.3.1), the tensor category $T_{\mathcal{A}}(\,\mathrm{Res}(\mathcal{A})\,)$ represents the $(m+1)$-Calabi-Yau completion of $\mathcal{A}$. Therefore, the localized $(m+1)$-Calabi-Yau completion is indeed a localization of the $(m+1)$-Calabi-Yau completion. Moreover, it is easy to see that the quasi-equivalence type of the localized $(m+1)$-Calabi-Yau completion only depends on the quasi-equivalence type of $\mathcal{A}$.

Now if $\mathcal{P}^{\mathrm{loc}}$ is formed by inverted some degree 0 generators in a finitely Sullivan DG category $\mathcal{P}$, then by Proposition (22.5.5), the canonical map $i : \mathcal{P} \to \mathcal{P}^{\mathrm{loc}}$ induces $i_!(\mathrm{Res}(\mathcal{P})) \cong \mathrm{Res}(\mathcal{P}^{\mathrm{loc}})$, and hence $\boldsymbol{L}i_!(\mathcal{P}) \simeq \mathcal{P}^{\mathrm{loc}}$. Since the functor $\boldsymbol{L}i_! : \mathcal{D}(\mathcal{P}^{\mathrm{e}}) \to \mathcal{D}((\mathcal{P}^{\mathrm{loc}})^{\mathrm{e}})$ preserves duality between perfect bimodules, any Calabi-Yau structure on $\mathcal{P}$ determines a Calabi-Yau structure on $\mathcal{P}^{\mathrm{loc}}$. In particular, for $\mathcal{P} = \Pi_{m+1}(\mathcal{A})$, this shows that the localized $(m+1)$-Calabi-Yau completion $\Pi^{\mathrm{loc}}_{m+1}(\mathcal{A})$ has a canonical $(m+1)$-Calabi-Yau structure.

As in the last subsection, one can use Theorem 24.2.5 and Proposition 24.3.3

to show that

$$\begin{aligned}
\Pi^{\mathrm{loc}}_{m+1}(\mathcal{A}) \;&=\; \mathrm{colim}\,[\,\mathcal{A} \xleftarrow{(\mathrm{id},\mathrm{id})} \mathcal{A} \amalg \mathcal{A}' \xrightarrow{(i,i')} T_{\mathcal{A}\amalg\mathcal{A}'}(\mathrm{Res}(\mathcal{A}))[\,E_x^{-1}\,]_{x\in R}\,] \\
&\simeq \mathrm{hocolim}\,[\,\mathcal{A} \xleftarrow{(\mathrm{id},\mathrm{id})} \mathcal{A} \amalg \mathcal{A}' \xrightarrow{(i,i')} \mathcal{A}\otimes I\,] \\
&\simeq \mathcal{A} \otimes \mathrm{hocolim}\,[\,* \xleftarrow{(\mathrm{id},\mathrm{id})} * \amalg * \xrightarrow{(i,i')} I\,] \\
&\simeq \mathcal{A} \otimes k\langle z^{\pm}\rangle
\end{aligned} \tag{24.4.2}$$

Moreover, if we define the *fundamental class* of the associative algebra $k\langle z^{\pm}\rangle$ to be the element $B(z) = 1 \otimes sz$ in the reduced bar complex of $k\langle z^{\pm}\rangle$ (cf. (24.3.7) and Definition (24.3.8)), then we have the following analogue of Proposition 24.3.9

Proposition 24.4.3. *Under the quasi-equivalence* (24.4.2), *the canonical* $(m+1)$-*Calabi-Yau structure on the localized* $(m+1)$-*Calabi-Yau completion* $\Pi^{\mathrm{loc}}_{m+1}(\mathcal{A})$ *corresponds to the class*

$$\mathsf{Sh}^{-}([\xi_{\mathcal{A}}] \otimes [B(z)]) \;\in\; \mathrm{HC}^{-}_{m+1}(\mathcal{A} \otimes k\langle z\rangle)$$

given by the shuffle product (22.4.13) *of the given* m-*Calabi-Yau structure* $[\xi_{\mathcal{A}}] \in \mathrm{HC}^{-}_{m}(\mathcal{A})$ *on* $\mathcal{A}$ *with the fundamental class* $[B(z)] \in \mathrm{HC}^{-}_{1}(k\langle z^{\pm}\rangle)$ *on* $k\langle z^{\pm}\rangle$.

Now we will investigate the effect of localization on relative Calabi-Yau completion. Thus, suppose $F : \mathcal{A} \to \mathcal{B}$ is a DG functor between finitely Sullivan DG categories $\mathcal{A} = T_{R_1}(Q_1)$ and $\mathcal{B} = T_{R_2}(Q_2)$, and suppose that $\mathcal{A}$ has a given m-Calabi-Yau structure $[\xi_{\mathcal{A}}] \in \mathrm{HC}^{-}_{m}(\mathcal{A})$.

The underlying weak m-Calabi-Yau structure $[\xi^{(0)}_{\mathcal{A}}] \in \mathrm{HH}_m(\mathcal{A})$ then induces a quasi-isomorphism $\hat{\xi}^{(0)}_{\mathcal{A}} s^{-m} : \mathrm{Res}(\mathcal{A})^{\vee}[m] \xrightarrow{\sim} \mathrm{Res}(\mathcal{A})$. Consider the composition

$$\mathrm{Res}(\mathcal{B})^{\vee}[m] \xrightarrow{(-1)^m s^m \gamma_F^{\vee} s^{-m}} F_!\mathrm{Res}(\mathcal{A})^{\vee}[m] \xrightarrow{F_!(\hat{\xi}^{(0)}_{\mathcal{A}} s^{-m})} F_!\mathrm{Res}(\mathcal{A}) \tag{24.4.4}$$

and let Ξ be the cone

$$\Xi := \operatorname{cone}\left[\operatorname{Res}(\mathcal{B})^\vee[m] \xrightarrow{(24.4.4)} F_!\operatorname{Res}(\mathcal{A})\right] \tag{24.4.5}$$

then it is clear that the canonical map

$$\tilde{F} : T_{\mathcal{A}}(\operatorname{Res}(\mathcal{A})) \to T_{\mathcal{B}}(\Xi) \tag{24.4.6}$$

gives an alternative model for the relative $(m+2)$-Calabi-Yau completion $\Pi_{m+1}(\mathcal{A}) \to \Pi_{m+2}(\mathcal{B})$. This allows one to make the following

Definition 24.4.7. The *localized relative* $(m+2)$-*Calabi-Yau completion* of $F : \mathcal{A} \to \mathcal{B}$ is defined to be the following localization of (24.4.6)

$$\tilde{F} : T_{\mathcal{A}}(\operatorname{Res}(\mathcal{A}))[\, E_x^{-1}\,]_{x\in R_1} \to T_{\mathcal{B}}(\Xi)[\, E_x^{-1}\,]_{x\in R_1}$$

and is denoted as $\tilde{F} : \Pi^{\mathrm{loc}}_{m+1}(\mathcal{A}) \to \Pi^{\mathrm{loc}}_{m+2}(\mathcal{B},\mathcal{A})$.

The discussion following Defintion 24.4.1 again shows that the localized relative $(m+2)$-Calabi-Yau completion $\tilde{F} : \Pi^{\mathrm{loc}}_{m+1}(\mathcal{A}) \to \Pi^{\mathrm{loc}}_{m+2}(\mathcal{B},\mathcal{A})$ has a canonical relative $(m+2)$-Calabi-Yau structure. The same is true when we consider deformations of the localized relative $(m+2)$-Calabi-Yau completion by negative cyclic classes. However, the deformation parameters in the localized case is different from those in the unlocalized case.

Recall that, given a deformation parameter $[\eta_{\mathcal{A}}] \in \mathrm{HH}_{m-1}(\mathcal{A})$, the deformed $(m+1)$-Calabi-Yau completion $\Pi_{m+1}(\mathcal{A},\eta_{\mathcal{A}})$ is independent of the choice $\eta_{\mathcal{A}}$ of representative of $[\eta_{\mathcal{A}}]$, because any other choice $\eta_{\mathcal{A}}+d(\zeta_{\mathcal{A}})$ determines an isomorphism (23.2.11). (The integer m in (23.2.11) should read as $m+1$ here. Therefore, the bimodule $M_{\mathcal{A}}^\vee[m-1]$ in (23.2.11) can be identified with $\operatorname{Res}(\mathcal{A})$ in the present context.)

However, if one localizes at the generators E_x of $T_{\mathcal{A}}(\mathrm{Res}(\mathcal{A}))$, the map (23.2.11) may not preserve this set of generators, and thus may not descend to a map on the localizations. This shows that the Hochschild homology classes $\mathrm{HH}_{m-1}(\mathcal{A})$ do not give well-defined deformation parameters of the localized $(m+1)$-Calabi-Yau completion (Definition 24.4.1).

In particular, in the relative case (Definition 24.4.7), the relative Hochschild homology classes $\mathrm{HH}_m(\mathcal{B}, \mathcal{A})$ also do not give well-defined deformation parameters. However, in this case, since we only localize the generators E_x in $T_{\mathcal{A}}(\mathrm{Res}(\mathcal{A}))$, one has well-defined deformations provided that one restricts to deformation parameters of the form

$$\eta \;=\; (0, \eta_{\mathcal{B}}) \;\in\; C_m(\mathcal{B}, \mathcal{A})$$

In other words, in the localized case, we consider deformation parameters $[\eta_{\mathcal{B}}] \in \mathrm{HH}_m(\mathcal{B})$, the corresponding map, denoted as

$$\tilde{F} : \Pi^{\mathrm{loc}}_{m+1}(\mathcal{A}) \;\to\; \Pi^{\mathrm{loc}}_{m+2}(\mathcal{B}, \mathcal{A}; \eta_{\mathcal{B}}) \tag{24.4.8}$$

will be called the *deformed localized relative* $(m+2)$-*Calabi-Yau completion* of the DG functor $F : \mathcal{A} \to \mathcal{B}$.

Deformation classes that were zero in the unlocalized case maybe become nonzero in the localized case. For example, let $[\xi_{\mathcal{A}}] \in \mathrm{HC}^{-}_m(\mathcal{A})$ be the given m-Calabi-Yau structure on $\mathcal{A}$, then its image $\gamma_F([\xi_{\mathcal{A}}]) \in \mathrm{HC}^{-}_m(\mathcal{B})$ under the induced map $\gamma_F : \mathrm{HC}^{-}_m(\mathcal{A}) \to \mathrm{HC}^{-}_m(\mathcal{B})$ gives a deformation parameter for the localized relative $(m+2)$-Calabi-Yau completion. This class is zero when passed to the relative negative cyclic homology $\mathrm{HC}^{-}_m(\mathcal{B}, \mathcal{A})$, but is nonzero in the localized case, and gives a deformation

$$\tilde{F} : \Pi^{\mathrm{loc}}_{m+1}(\mathcal{A}) \;\to\; \Pi^{\mathrm{loc}}_{m+2}(\mathcal{B}, \mathcal{A}; \gamma_F(\xi_{\mathcal{A}})) \tag{24.4.9}$$

Definition 24.4.10. The map (24.4.9) is called the *canonically deformed localized relative* $(m+2)$*-Calabi-Yau completion* of the DG functor $F : \mathcal{A} \to \mathcal{B}$.

CHAPTER 25

THE PERVERSELY THICKENED DG CATEGORY

In this section, we give the main construction, called the perversely thickened DG category, associated to pairs (N, M) consisting of a manifold N and an embedded submanifold M of codimension ≥ 2. We calculate this DG category in two examples: points in the 2-dimensional disk, and links in $\mathbb{R}^3$. Both example give DG categories of interest in contact geometry. We also give a relation with perverse sheaves.

25.1 Adams-Hilton models

In this subsection, we recall some constructions in [2, 3, 89, 46, 80], which allows one to model topological spaces by DG categories. All topological spaces in Part IV of this thesis are assumed to be compactly generated and weakly Hausdorff.

First, recall that there is a Quillen equivalence

$$|-| : \mathtt{sSet} \rightleftarrows \mathtt{Top} : S(-)$$

between the model category sSet of simplicial sets and the model category Top of topological spaces, where $|-|$ is the geometric realization functor and $S(-)$ is the singular functor. (See, e.g., [67])

If X is connected and pointed, then there is a simplicial subset $ES(X) \subset S(X)$, called the *first Eilenberg subcomplex*, whose set $ES_n(X)$ at simplicial degree n is defined to be the set of continuous maps $|\Delta^n| \to X$ such that each vertex is mapped to the basepoint $* \in X$. The inclusion $ES(X) \hookrightarrow S(X)$ is a weak equivalence. *i.e.,* it induces a homotopy equivalence on their geometric realization $|ES(X)| \hookrightarrow |S(X)|$.

The Eilenberg subcomplex gives a functor $ES(-) : \mathtt{Top}_0^* \to \mathtt{sSet}_0$ from the category $\mathtt{Top}_0^*$ of pointed connected topological spaces to the category $\mathtt{sSet}_0$ of reduced simplicial sets, where a simplicial set $X \in \mathtt{sSet}$ is said to be *reduced* if its set X_0 at simplicial degree 0 is a singleton. The functor $ES(-)$ is a right Quillen functor, with left adjoint given by the geometric realization functor $|-| : \mathtt{sSet}_0 \to \mathtt{Top}_0^*$. This Quillen pair

$$|-| \,:\, \mathtt{sSet}_0 \rightleftarrows \mathtt{Top}_0^* \,:\, ES(-) \tag{25.1.1}$$

is in fact a Quillen equivalence pair, and hence induce mutually inverse equivalences of homotopy categories $\mathrm{Ho}(\mathtt{sSet}_0) \simeq \mathrm{Ho}(\mathtt{Top}_0^*)$. This allows one to model connected pointed topological spaces by reduced simplicial sets.

Now, given a reduced simplicial set $X \in \mathtt{sSet}_0$, Kan has constructed in [89] a simplicial group $\mathbb{G}(X) \in \mathtt{sGr}$, called the *Kan loop group*, which models the based (Moore) loop space of the topological space $|X| \in \mathtt{Top}_0^*$. (see, e.g., [112, 67]) The functor $\mathbb{G}(-) : \mathtt{sSet}_0 \to \mathtt{sGr}$ has a right adjoint $\overline{W} : \mathtt{sGr} \to \mathtt{sSet}_0$. The adjunction

$$\mathbb{G}(-) \,:\, \mathtt{sSet}_0 \rightleftarrows \mathtt{sGr} \,:\, \overline{W} \tag{25.1.2}$$

forms Quillen equivalence pair (see [67, Proposition V.6.3]).

The composition of equivalences of homotopy categories

$$\mathrm{Ho}(\mathtt{Top}_0^*) \xrightarrow{ES(-)} \mathrm{Ho}(\mathtt{sSet}_0) \xrightarrow{\mathbb{G}(-)} \mathrm{Ho}(\mathtt{sGr})$$

allows one to model topological spaces by simplicial groups.

Recall that, for every simplicial commutative algebra A, its Dold-Kan normalization $N(A)$ inherits the structure of a DG algebra. In fact, the functor

$N : \mathtt{sAlg}_k \to \mathtt{DGA}_k^+$ is part of a Quillen equivalence pair (see [145])

$$N^* : \mathtt{DGA}_k^+ \rightleftarrows \mathtt{sAlg}_k : N$$

We call this Quillen equivalence pair the *monoidal Dold-Kan correspondence*.

Suppose we are given a simplicial group Γ that corresponds to a topological space X under the Quillen equivalences (25.1.2) and (25.1.1). Then applying the group algebra functor degreewise gives a simplicial associative algebra $k[\Gamma] \in \mathtt{sAlg}_k$. By monoidal Dold-Kan correspondence, the normalization $N(k[\Gamma])$ inherits the structure of a DG algebra. This DG algebra is quasi-isomorphic to the chain algebra of the Moore loop space of X.

Definition 25.1.3. The image of a connected pointed topological space X under the composition

$$\mathrm{Ho}(\mathtt{Top}_0^*) \xrightarrow{ES(-)} \mathrm{Ho}(\mathtt{sSet}_0) \xrightarrow{\mathbb{G}(-)} \mathrm{Ho}(\mathtt{sGr}) \xrightarrow{k[-]} \mathrm{Ho}(\mathtt{sAlg}_k) \xrightarrow{N(-)} \mathrm{Ho}(\mathtt{DGA}_k^+)$$

is called the *Adams-Hilton model* of X, and is denoted as $\mathsf{AH}(X)$.

Despite the apparently complicated definition, the Adams-Hilton model admit simple descriptions in many cases. For example, when the space X is simply connected, one can find a 1-reduced simplicial set, still denoted by X, which represents the homotopy type of X. Then, the Adams-Hilton model of X is simply given by the cobar construction of the DG coalgebra $C_*(X)$. *i.e.*, we have

$$\mathsf{AH}(X) \simeq \Omega(C_*(X))$$

This justifies the name 'Adams-Hilton model' because this construction was given by J . F. Adams and P. Hilton [2, 3].

More generally, the cobar construction of Adams and Hilton can be extended to arbitrary reduced simplcial set $X \in \mathtt{sSet}_0$. In this case, the non-degenerate simplices of X of degree 1 will contribute to degree 0 generators in the cobar cosntruction $\Omega(C_*(X))$. It is shown in [80] that, if we invert these generators, then the resulting DG algebra is the Adams-Hilton model of X. We write this as

$$\mathsf{AH}(X) \simeq \Omega^{\mathrm{loc}}(C_*(X)) \tag{25.1.4}$$

In particular, consider the $(r+1)$-dimensional sphere S^{r+1}, for $r \geq 0$. If we take X to be the reduced simplicial set with only two non-degenerate simplices, one in degree 0 and one in degree $r+1$, then by (25.1.4), the Adams-Hilton model of S^{r+1} is given by

$$\mathsf{AH}(S^{r+1}) = \begin{cases} k\langle z \rangle, & |z| = r, \qquad \text{if } r > 0 \\ k\langle z^{\pm} \rangle, & |z| = 0, \qquad \text{if } r = 0 \end{cases} \tag{25.1.5}$$

Recall that every CW complex X can be written as the direct limit of a sequence of homotopy pushout

$$\begin{tikzcd} \amalg_{\alpha \in I_r} S^r_\alpha \arrow[r] \arrow[d] & \amalg_{\alpha \in I_r} D^{r+1}_\alpha \arrow[d] \\ sk_r(X) \arrow[r] & sk_{r+1}(X) \end{tikzcd}$$

Since every functor in Definition 25.1.3 preserve homotopy colimits, one can also write the Adams-Hilton model $\mathsf{AH}(X)$ of X as the direct limit of such a successive sequence of homotopy pushout. Combined with (25.1.5), this can be used to show (see [81]) that, for a reduced CW complex X (*i.e.*, its 0-th skeleton $sk_0(X)$ is a singleton $\{*\}$), its Adams-Hilton model is almost semi-free, of the form

$$\mathsf{AH}(X) = k\langle x_1^{\pm}, x_2^{\pm}, \ldots, y_1, y_2, \ldots \rangle \tag{25.1.6}$$

where the generators x_i are of degree 0, and the generators y_i are of positive degree. Moreover, for $r \geq 0$, the generators in (25.1.6) of degree r are in bijective correspondence with the CW cells of X of degree $r+1$.

The notion of Adams-Hilton model can be extended to topological spaces that are not necessarily connected. To do this, one replaces Kan's loop group functor $\mathbb{G}(-)$ with the *Dwyer-Kan loop groupoid functor* constructed in [46]. The Dwyer-Kan loop groupoid functor is a functor $\mathbb{G}(-) : \mathtt{sSet} \to \mathtt{sGrpd}$ from the category of (not necessarily reduced) simplicial sets to the category of simplicially enriched groupoids. It has a right adjoint, which form a Quillen equivalence pair

$$\mathbb{G}(-) \,:\, \mathtt{sSet} \rightleftarrows \mathtt{sGrpd} \,:\, \overline{W} \tag{25.1.7}$$

Applying the k-linearization functor $k[-]$ to a simplicial groupoid Γ gives a category $k[\Gamma] \in \mathtt{Cat}_{\mathtt{sMod}_k}$ enriched over simplicial k-modules. Then, applying the Dold-Kan normalization $N(-)$ to each Hom-complexes in $k[\Gamma]$ gives a non-negatively graded DG category $N(k[\Gamma]) \in \mathtt{dgCat}_{\geq 0}$ (see, e.g., [153]). These construction allows one to define Adams-Hilton models for topological spaces that are not necessarily connected.

Definition 25.1.8. The image of a topological space X under the composition

$$\mathrm{Ho}(\mathtt{Top}) \xrightarrow{S(-)} \mathrm{Ho}(\mathtt{sSet}) \xrightarrow{\mathbb{G}(-)} \mathrm{Ho}(\mathtt{sGrpd}) \xrightarrow{k[-]} \mathrm{Ho}(\mathtt{Cat}_{\mathtt{sMod}_k}) \xrightarrow{N(-)} \mathrm{Ho}(\mathtt{dgCat}_{\geq 0})$$

is called the *Adams-Hilton model* of X, and is denoted as $\mathsf{AH}(X)$.

In practice, the Adams-Hilton model of any topological space X can be taken to be the disjoint union of the Adams-Hilton DGA models of the path components of X.

25.2 Calabi-Yau structures on Adams-Hilton models

In [15], Brav and Dyckerhoff constructed a map

$$\alpha : \mathrm{H}_*(X) \to \mathrm{HC}^-_*(\mathsf{AH}(X)) \tag{25.2.1}$$

from the homology $\mathrm{H}_*(X)$ of any topological space X to the negative cyclic homology $\mathrm{HC}^-_*(\mathsf{AH}(X))$ of its Adams-Hilton model.

This map connects the topology of X with the noncommutative geometry of its Adams-Hilton model. In particular, Poincaré duality on X translates to Calabi-Yau structures on $\mathsf{AH}(X)$. More precisely, let X be a closed oriented manifold of dimension n, and let $[\xi_X] \in \mathrm{HC}^-_n(\mathsf{AH}(X))$ be the image under (25.2.1) of the fundamental class $[X] \in \mathrm{H}_*(X)$. Abusing the terminology, we will still call $[\xi_X]$ the *fundamental class* of X. Then we have

Theorem 25.2.2 ([15])**.** *The fundamental class $[\xi_X] \in \mathrm{HC}^-_n(\mathsf{AH}(X))$ is an n-Calabi-Yau structure on $\mathsf{AH}(X)$.*

This theorem extends to the case of manifold with boundary. Thus, let X be a compact oriented manifold of dimension n, with a boundary ∂X. Then the inclusion $\partial X \hookrightarrow X$ induces a map $F : \mathsf{AH}(\partial X) \to \mathsf{AH}(X)$, called the *peripheral map*, on their Adams-Hilton models. By functoriality of the map (25.2.1), the induced map on the cone gives

$$\alpha : \mathrm{H}_*(X, \partial X) \to \mathrm{HC}^-_*(\mathsf{AH}(X), \mathsf{AH}(\partial X)) \tag{25.2.3}$$

Denote by $[\xi_X] \in \mathrm{HC}^-_n(\mathsf{AH}(X), \mathsf{AH}(\partial X))$ the image under (25.2.3) of the relative fundamental class $[X] \in \mathrm{H}_n(X, \partial X)$, and still call $[\xi_X]$ the *relative fundamental class* of X. Then we have

Theorem 25.2.4 ([15]). *The relative fundamental class $[\xi_X] \in \mathrm{HC}^-_n(\mathsf{AH}(X), \mathsf{AH}(\partial X))$ is an n-Calabi-Yau structure on the peripheral map $F : \mathsf{AH}(\partial X) \to \mathsf{AH}(X)$.*

Now we consider Calabi-Yau completions of Adams-Hilton models of manifolds. Suppose that M is a closed oriented manifold of dimension m, then its Adams-Hilton model $\mathsf{AH}(M)$ has a canonical m-Calabi-Yau structure $[\xi_M] \in \mathrm{HC}^-_m(\mathsf{AH}(M))$. We can apply the results of Sections 24.3 and 24.4 to study the Calabi-Yau completion of $\mathsf{AH}(M)$. We summarize this into a theorem.

Theorem 25.2.5. *For $p > 0$, the $(m+p+1)$-Calabi-Yau completion of $\mathsf{AH}(M)$ is the Adams-Hilton model of $M \times S^{p+1}$. i.e., we have*

$$\Pi_{m+p+1}(\mathsf{AH}(M)) \simeq \mathsf{AH}(M \times S^{p+1}) \tag{25.2.6}$$

For $p = 0$. the localized $(m+1)$-Calabi-Yau completion of $\mathsf{AH}(M)$ is the Adams-Hilton model of $M \times S^1$. i.e., we have

$$\Pi^{\mathrm{loc}}_{m+1}(\mathsf{AH}(M)) \simeq \mathsf{AH}(M \times S^1) \tag{25.2.7}$$

Moreover, under the identifications (25.2.6) *and* (25.2.7)*, the canonical $(m+p+1)$-Calabi-Yau structures on these (localized) completions correspond to the fundamental class on Adams-Hilton model of $M \times S^{p+1}$.*

Proof. By (24.3.6), the $(m+p+1)$-Calabi-Yau completion of $\mathsf{AH}(M)$ is given by the tensor product

$$\Pi_{m+p+1}(\mathsf{AH}(M)) \simeq \mathsf{AH}(M) \otimes k\langle z \rangle \tag{25.2.8}$$

where z has degree $|z| = p$.

If $p > 0$, then by (25.1.5), the DG algebra $k\langle z\rangle$ appearing above is the Adams-Hilton model of the $(p+1)$-sphere.

Now, notice that the Adams-Hilton model of a product of spaces is simply the derived tensor products of their Adams-Hilton models, *i.e.*,

$$\mathsf{AH}(X \times Y) \simeq \mathsf{AH}(X) \otimes^{\boldsymbol{L}} \mathsf{AH}(Y) \tag{25.2.9}$$

Combining these results then give (25.2.6) when $p > 0$.

When $p = 0$, one can similarly combine (24.4.2), (25.1.5) and (25.2.9) to show (25.2.7).

The last statement then follows from Proposition 24.3.9 and 24.4.3. □

25.3 Calabi-Yau cospans and gluing

As suggested in the introduction, relative Calabi-Yau structures can be 'glued along common boundaries'. To formulate this, one thinks of a relative Calabi-Yau structure on a DG functor $F : \mathcal{A} \to \mathcal{B}$ as a noncommtative analogoue of a manifold with boundary, where $\mathcal{A}$ plays the role of the boundary. This interpretation is natural in view of Theorem 25.2.4 Then, one can imitate the category of cobordism. This leads us to consider cospans of DG categories.

A *cospan* of DG categories is a DG functor $(F_2, F_1) : \mathcal{A}_2 \amalg \mathcal{A}_1 \to \mathcal{X}$. This can be written alternatively in the form $[\,\mathcal{A}_2 \xrightarrow{F_2} \mathcal{X} \xleftarrow{F_1} \mathcal{A}_1\,]$. We think of this as a cobordism from $\mathcal{A}_1$ to $\mathcal{A}_2$. This suggests a notion of *composition of cospans*. Thus, if we are given two cospans

$$(G_3, G_2) : \mathcal{A}_3 \amalg \mathcal{A}_2 \to \mathcal{Y} \qquad \text{and} \qquad (F_2, F_1) : \mathcal{A}_2 \amalg \mathcal{A}_1 \to \mathcal{X} \tag{25.3.1}$$

then their composition is defined to be the cospan

$$[\,\mathcal{A}_3 \xrightarrow{G_3} \mathcal{Y} \xleftarrow{G_2} \mathcal{A}_2\,] \circ [\,\mathcal{A}_2 \xrightarrow{F_2} \mathcal{X} \xleftarrow{F_1} \mathcal{A}_1\,] := [\,\mathcal{A}_3 \xrightarrow{H_3} \mathcal{Y} \amalg^{\boldsymbol{L}}_{\mathcal{A}_2} \mathcal{X} \xleftarrow{H_1} \mathcal{A}_1\,] \tag{25.3.2}$$

where H_1 and H_3 are the obvious maps to the homotopy pushout $\mathcal{Y} \amalg^{\boldsymbol{L}}_{\mathcal{A}_2} \mathcal{X}$.

The analogy with cobordism of manifolds suggests that, if both the cospan $(G_3, G_2) : \mathcal{A}_3 \amalg \mathcal{A}_2 \to \mathcal{Y}$ and $(F_2, F_1) : \mathcal{A}_2 \amalg \mathcal{A}_1 \to \mathcal{X}$ have relative n-Calabi-Yau structures that coincides in some sense on the 'common boundary' $\mathcal{A}_2$, then one should be able to glue these to a Calabi-Yau structure on the composition $(H_3, H_1) : \mathcal{A}_3 \amalg \mathcal{A}_1 \to \mathcal{Y} \amalg^{\boldsymbol{L}}_{\mathcal{A}_2} \mathcal{X}$.

This is indeed true, and is worked out in [15]. More precisely, consider the canonical maps of mixed complexes

$$C_*(\,\mathcal{X}\,,\,\mathcal{A}_2 \amalg \mathcal{A}_1\,) \to C_*(\mathcal{A}_2)[1] \qquad \text{and} \qquad C_*(\,\mathcal{Y}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_2\,) \to C_*(\mathcal{A}_2)[1]$$

This allows one to define the following mixed complex

$$C_*(\,\mathcal{Y}/\mathcal{A}\backslash\mathcal{X}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_1\,) := \operatorname{cocone}\,[\,C_*(\,\mathcal{Y}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_2\,) \oplus C_*(\,\mathcal{X}\,,\,\mathcal{A}_2 \amalg \mathcal{A}_1\,) \to C_*(\mathcal{A}_2)[1]\,] \tag{25.3.3}$$

By definition, this mixed complex has a canonical maps to the relative Hochschild complexes of both of the cospans (25.3.1)

$$\begin{aligned} \pi_{\mathcal{Y}} &: C_*(\,\mathcal{Y}/\mathcal{A}\backslash\mathcal{X}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_1\,) \to C_*(\,\mathcal{Y}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_2\,) \\ \pi_{\mathcal{X}} &: C_*(\,\mathcal{Y}/\mathcal{A}\backslash\mathcal{X}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_1\,) \to C_*(\,\mathcal{X}\,,\,\mathcal{A}_2 \amalg \mathcal{A}_1\,) \end{aligned} \tag{25.3.4}$$

Moreover, one has a map of mixed complexes (see [15])

$$\chi : C_*(\,\mathcal{Y}/\mathcal{A}\backslash\mathcal{X}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_1\,) \to C_*(\,\mathcal{Y} \amalg^{\boldsymbol{L}}_{\mathcal{A}} \mathcal{X}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_1\,) \tag{25.3.5}$$

Denote by $\mathrm{HC}^-_*(\,\mathcal{Y}/\mathcal{A}\backslash\mathcal{X}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_1\,)$ the negative cyclic homology of the mixed complex (25.3.3). Then the maps (25.3.4) and (25.3.5) allow one to formulate the following theorem proved in [15].

Theorem 25.3.6 ([15]). *Suppose that* $[\xi] \in \mathrm{HC}_n^-(\,\mathcal{Y}/\mathcal{A}\backslash\mathcal{X}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_1\,)$ *is a class whose projections* $\pi_{\mathcal{X}}([\xi]) \in \mathrm{HC}_n^-(\,\mathcal{X}\,,\,\mathcal{A}_2 \amalg \mathcal{A}_1\,)$ *and* $\pi_{\mathcal{Y}}([\xi]) \in \mathrm{HC}_n^-(\,\mathcal{Y}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_2\,)$ *are relative n-Calabi-Yau structures on both of the cospans* (25.3.1)*, then the image* $\chi([\xi]) \in \mathrm{HC}_n^-(\,\mathcal{Y} \amalg^{\boldsymbol{L}}_{\mathcal{A}} \mathcal{X}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_1\,)$ *under* (25.3.5) *gives a relative n-Calabi-Yau structure on the composition* (25.3.2).

This theorem discusses gluing of Calabi-Yau structures at the level of chain complexes. This implies a (non-canonical) gluing result at the level of homology.

Corollary 25.3.7. *Suppose that* $[\xi_{\mathcal{X}}] \in \mathrm{HC}_n^-(\,\mathcal{X}\,,\,\mathcal{A}_2 \amalg \mathcal{A}_1\,)$ *and* $[\xi_{\mathcal{Y}}] \in \mathrm{HC}_n^-(\,\mathcal{Y}\,,\,\mathcal{A}_3 \amalg \mathcal{A}_2\,)$ *are relative n-Calabi-Yau structures on the cospans* (25.3.1)*, whose coboundaries coincide on* $\mathrm{HC}_{n-1}^-(\mathcal{A}_2)$*, i.e., we have*

$$(\delta([\xi_{\mathcal{X}}]))|_{\mathcal{A}_2} = (\delta([\xi_{\mathcal{Y}}]))|_{\mathcal{A}_2}$$

then there is a relative n-Calabi-Yau structure on the composition (25.3.2)*. Moreover, this structure is canonical if* $\mathrm{HC}_n^-(\mathcal{A}_2) = 0$.

Proof. Apply the long exact sequence on negative cyclic homology associated to the distinguished triangle defining the cocone (25.3.3). This shows the existence of a class $[\xi]$ satisfying the assumptions of Theorem 25.3.6. Applying the theorem then proves the claim. $\square$

25.4 The perversely thickened DG category

Let N be a compact oriented manifold of dimension n, possibly with a boundary ∂N, and let M be a closed oriented manifold of dimension m smoothly embedded in the interior of N, with an open tubular neighborhood $\nu(M) \subset N$. We only consider the case where M has codimension $n - m \geq 2$, and let $p := n - m - 2$.

Let $X = N \setminus \nu(M)$ be the complement of this tubular neighborhood. Then its boundary ∂X is the disjoint union of two parts $\partial X = \partial N \cup S_M(N)$, where $S_M(N)$ is the total space of the unit conormal bundle $\pi : S_M(N) \to M$, which is an S^{p+1}-bundle over M. Suppose that this S^{m+1}-bundle is trivial. We call a trivialization $\Phi : M \times S^{m+1} \xrightarrow{\sim} S_M(N)$ a *normal framing* of $M \hookrightarrow N$.

As explained in the introduction, we will produce a DG category $\mathscr{A}(N, M; \Phi)$ by a gluing construction. First, let $\mathsf{AH}(M)$ be the Adams-Hilton model of the manifold M. Consider the directed cylinder over $\mathsf{AH}(M)$

$$\mathcal{A} := \mathsf{AH}(M) \amalg \mathsf{AH}(M)' \xrightarrow{F} \mathsf{AH}(M) \otimes \vec{I} =: \mathcal{B} \tag{25.4.1}$$

where we have notationally distinguished the second identical copy $\mathsf{AH}(M)'$ from the first copy $\mathsf{AH}(M)$.

Notice that, since M is a closed oriented manifold of dimension m, it has a canonical m-Calabi-Yau structure $[\xi_M] \in \mathrm{HC}^-_m(\mathsf{AH}(M))$. This induces an (m)-Calabi-Yau structure on $\mathcal{A}$, given by

$$[\xi_{\mathcal{A}}] := (\,[\xi_M]\,,\, -[\xi_M]\,) \in \mathrm{HC}^-_m(\mathsf{AH}(M)) \oplus \mathrm{HC}^-_m(\mathsf{AH}(M)') \cong \mathrm{HC}^-_m(\mathcal{A}) \tag{25.4.2}$$

This allows one to apply the results in Section 24.3 and 24.4 to the relative Calabi-Yau completion of (25.4.1).

Definition 25.4.3. The *n-dimensional perverse neighborhood* of M is the DG category $\mathscr{I}_n(M)$ defined as follows

1. If $p > 0$, then the perverse neighborhood is the relative Calabi-Yau completion

$$\mathscr{I}_n(M) := \Pi_n(\mathcal{B}, \mathcal{A})$$

2. If $p = 0$, then the perverse neighborhood is the canonically deformed localized relative Calabi-Yau completion (see Definition 24.4.10)

$$\mathscr{I}_n(M) := \Pi_n^{\mathrm{loc}}(\mathcal{B}, \mathcal{A}; \gamma_F(\xi_{\mathcal{A}}))$$

By definition of (localized) relative Calabi-Yau completions, there come with canonical maps

$$\Pi_{n-1}(\mathcal{A}) = \left(\Pi_{n-1}(\mathsf{AH}(M))\right) \amalg \left(\Pi_{n-1}(\mathsf{AH}(M)')\right) \to \Pi_n(\mathcal{B}, \mathcal{A})$$
$$\Pi_{n-1}^{\mathrm{loc}}(\mathcal{A}) = \left(\Pi_{n-1}^{\mathrm{loc}}(\mathsf{AH}(M))\right) \amalg \left(\Pi_{n-1}^{\mathrm{loc}}(\mathsf{AH}(M)')\right) \to \Pi_n^{\mathrm{loc}}(\mathcal{B}, \mathcal{A})$$

By Theorem 25.2.5, the domain of these maps can be identified with the Adams-Hilton models of $M \times S^{p+1}$. Therefore, there is a canonical map

$$\mathsf{AH}(M \times S^{p+1}) \amalg \mathsf{AH}(M \times S^{p+1})' \xrightarrow{(i,i')} \mathscr{I}_n(M) \tag{25.4.4}$$

This map allows us to glue this DG category with the topology of the embedding $M \hookrightarrow N$.

Thus, returning to the setting of the beginning of this subsection. Then the normal framing $\Phi : M \times S^{p+1} \xrightarrow{\sim} S_M(N)$ allows one to identify the Adams-Hilton models $\mathsf{AH}(M \times S^{p+1}) \simeq \mathsf{AH}(S_M(N))$. The decomposition of ∂X into the disjoint union $\partial X = \partial N \cup S_M(N)$ then allows us to give a cospan of DG categories

$$\mathsf{AH}(M \times S^{p+1}) \amalg \mathsf{AH}(\partial N) \xrightarrow{(\mathsf{AH}(\Phi), i_{\partial N})} \mathsf{AH}(X) \tag{25.4.5}$$

One can then form the composition of the cospans (25.4.4) and (25.4.5). The result is the following

Definition 25.4.6. The *perversely thickened DG category* of (N, M, Φ) is the DG category $\mathscr{A}(N, M; \Phi)$ defined as the homotopy pushout

$$\mathscr{A}(N, M; \Phi) := \operatorname{hocolim} \left[\mathscr{I}_n(M) \xleftarrow{i'} \mathsf{AH}(M \times S^{p+1}) \xrightarrow{\mathsf{AH}(\Phi)} \mathsf{AH}(X) \right]$$

This comes with a canonical map

$$\mathsf{AH}(M \times S^{p+1}) \amalg \mathsf{AH}(\partial N) \to \mathscr{A}(N, M; \Phi) \tag{25.4.7}$$

called the *perverse peripheral map*.

Then we have

Theorem 25.4.8. *The perverse peripheral map has a relative n-Calabi-Yau structure. Moreover, if M is aspherical, i.e., if $\pi_i(M) = 0$ for all $i > 1$, then this structure is canonical.*

Proof. The perverse peripheral map is the composition of the cospans (25.4.4) and (25.4.5). By Corollary 25.3.7, to give a relative n-Calabi-Yau structure, it suffices to give relative n-Calabi-Yau structures on both (25.4.4) and (25.4.5) whose induced $(n-1)$-Calabi-Yau structures on $\mathsf{AH}(M \times S^{p+1})$ agree.

By construction, the cospan (25.4.4) can be identified with a (localized) n-Calabi-Yau completion, and therefore has a canonical relative n-Calabi-Yau structure. Moreover, by Theorem 25.2.5, under this identification the induced $(n-1)$-Calabi-Yau structures on $\mathsf{AH}(M \times S^{p+1})$ is the fundamental class of the closed oriented manifold $M \times S^{p+1}$.

The other cospan (25.4.5) has a relative n-Calabi-Yau structure given by the relative fundamental class of X. The induced $(n-1)$-Calabi-Yau structures on $\mathsf{AH}(M \times S^{p+1})$ is therefore also given by the fundamental class of $M \times S^{p+1}$. These two classes therefore coincide in $\mathsf{AH}(M \times S^{p+1})$.

Finally, notice that if M is aspherical, then so is $M \times S^1$. Hence $\mathrm{HC}_n^-(M \times S^1) = 0$. The last statement then follows again from Corollary 25.3.7. $\square$

25.5 Multiplicative preprojective algebra

We compute the perversely thickened DG category for the case when N is the 2-dimensional disk $N = D$, and M is a finite set of points $M = \{p_1, \ldots, p_n\}$ in the interior of N. In this case, there is a unique trivialization of the conormal bundle of M in N. We will therefore not mention this trivialization in our notation.

We first compute the 2-dimensional perverse neighborhood of the 0-dimensional manifold M. Since M is a disjoint union of n points, it suffices to perform this calculation for the case when M has only one point $M = \{p_1\}$.

Thus, we first compute the deformed relative 2-Calabi-Yau completion of the map

$$F \,:\, \mathcal{A} \,=\, \big[\,1 \qquad 0\,\big] \,\hookrightarrow\, \big[\,1 \xrightarrow{a} 0\,\big] \,=:\, \mathcal{B} \tag{25.5.1}$$

where the arrow a has degree 0.

The standard resolution $\mathrm{Res}(\mathcal{A})$ of $\mathcal{A}$ is given by $\mathcal{A}$ itself, which is semi-free over the basis set $\{E_1, E_0\}$. The standard resolution $\mathrm{Res}(\mathcal{B})$ of $\mathcal{B}$ is semi-free over the basis set $\{E_1, E_0, sDa\}$ where the basis element sDa has differential

$$d(sDa) = a \cdot E_1 - E_0 \cdot a$$

Therefore, the cone

$$\Xi \,=\, \mathrm{cone}\,[\,\mathrm{Res}(\mathcal{B})^\vee \xrightarrow{\gamma_F^\vee} \mathrm{Res}(\mathcal{A})^\vee\,]$$

is semi-free over the basis

$$\{\, s(E_{1,\mathcal{B}})^\vee \,,\, s(E_{0,\mathcal{B}})^\vee \,,\, s(sDa)^\vee \,,\, (E_{1,\mathcal{A}})^\vee \,,\, (E_{0,\mathcal{A}})^\vee \,\} \tag{25.5.2}$$

with differentials (see (22.1.5) and (22.1.6)) given by

$$
\begin{aligned}
&d(s(sDa)^\vee) = 0 \qquad d((E_{1,\mathcal{A}})^\vee) = 0 \qquad d((E_{0,\mathcal{A}})^\vee) = 0 \\
&d(s(E_{1,\mathcal{B}})^\vee) = (E_{1,\mathcal{A}})^\vee + s(sDa)^\vee \cdot a \\
&d(s(E_{0,\mathcal{B}})^\vee) = (E_{0,\mathcal{A}})^\vee - a \cdot s(sDa)^\vee
\end{aligned}
\tag{25.5.3}
$$

Rename the basis elements (25.5.2) as

$$
\begin{aligned}
&\mu_1 := -(E_{1,\mathcal{A}})^\vee \qquad \mu_0 := (E_{0,\mathcal{A}})^\vee \\
&\xi_1 := -s(E_{1,\mathcal{B}})^\vee \qquad \xi_0 := s(E_{0,\mathcal{B}})^\vee \\
&a^* := s(sDa)^\vee
\end{aligned}
\tag{25.5.4}
$$

The relative Calabi-Yau completion of (25.5.1) is then has the presentation

$$
\Pi_1(\mathcal{A}) \;=\; k\left\langle
\begin{tikzcd}
1 \arrow["\mu_1", loop above] & & 0 \arrow["\mu_0", loop above]
\end{tikzcd}
\right\rangle
\tag{25.5.5}
$$

mapping into

$$
\Pi_2(\mathcal{B},\mathcal{A}) \;=\; k\left\langle
\begin{tikzcd}
1 \arrow["\mu_1", loop above] \arrow["\xi_1"', loop below] \arrow[rr, bend left=10, "a"] & & 0 \arrow["\mu_0", loop above] \arrow["\xi_0"', loop below] \arrow[ll, bend left=10, "a^*"]
\end{tikzcd}
\right\rangle
\tag{25.5.6}
$$

The deformation parameter for the canonically deformed localized relative 2-Calabi-Yau completion (Definition 24.4.10) is given by the image of (25.4.2) under the map $\gamma_F : \mathrm{HC}_0^-(\mathcal{A}) \to \mathrm{HC}_0^-(\mathcal{B})$. Therefore, it is represented by

$$
\gamma_F(\xi_{\mathcal{A}}) \;=\; E_{1,\mathcal{B}} - E_{0,\mathcal{B}} \quad \in \mathcal{B} \otimes_{\mathcal{B}^e} \mathrm{Res}(\mathcal{B})
$$

Therefore, the deformed localized relative 2-Calabi-Yau completion of (25.5.1)

$$
\Pi_1^{\mathrm{loc}}(\mathcal{A}) \;\to\; \Pi_2^{\mathrm{loc}}(\mathcal{B},\mathcal{A};\gamma_F(\xi_{\mathcal{A}}))
\tag{25.5.7}
$$

is given by the presentation

$$k\langle\mu_1^{\pm}\rangle \amalg k\langle\mu_0^{\pm}\rangle \longrightarrow k\left\langle \begin{tikzcd} \boxed{k\langle\mu_1^{\pm}\rangle} \arrow[loop above, "\xi_1"] \arrow[r, bend left, "a"] & \boxed{k\langle\mu_0^{\pm}\rangle} \arrow[loop above, "\xi_0"] \arrow[l, bend left, "a^*"] \end{tikzcd} \right\rangle \tag{25.5.8}$$

with differentials

$$\begin{aligned} d(\mu_1) &= d(\mu_0) = 0 \\ d(a) &= d(a^*) = 0 \\ d(\xi_1) &= \mu_1 - a^*a - 1 \\ d(\xi_0) &= \mu_0 - aa^* - 1 \end{aligned} \tag{25.5.9}$$

This gives a presentation of the 2-dimensional perverse neighborhood $\mathscr{I}_2(\{*\})$ of 0-dimensional manifold $\{*\}$.

The other ingredient of the perversely thickened DG category (Definition 25.4.6) of the pair $(D^2, \{p_1, \ldots, p_n\})$ is the Adams-Hilton model of the complement $X = D^2 \setminus \nu\{p_1, \ldots, p_n\}$ of the tubular neighborhood of $\{p_1, \ldots, p_n\}$ in the disk D^2. Since X is aspherical, its Adams-Hilton model $\mathsf{AH}(X)$ is simply given by the group algebra of its fundamental group

$$\mathsf{AH}(X) \;=\; k[\pi_1(X)] \;=\; k\langle T_1^{\pm}, \ldots, T_n^{\pm}\rangle$$

The peripheral maps are specified with

$$\mathsf{AH}(\partial D^2) \;=\; k\langle Q^{\pm}\rangle \xrightarrow{Q \mapsto T_1 \ldots T_n} k\langle T_1^{\pm}, \ldots, T_n^{\pm}\rangle \;=\; \mathsf{AH}(X) \tag{25.5.10}$$

$$\mathsf{AH}(\partial\nu\{p_i\}) \;=\; k\langle \mu_i'^{\pm}\rangle \xrightarrow{\mu_i' \mapsto T_i} k\langle T_1^{\pm}, \ldots, T_n^{\pm}\rangle \;=\; \mathsf{AH}(X) \tag{25.5.11}$$

Therefore, to compute the perversely thickened DG category $\mathscr{A}(D^2, \{p_1, \ldots, p_n\})$, we form the n-fold disjoint union of $\mathscr{I}_2(\{*\})$, and glue it with the Adams-Hilton model of the complement X. For notational convenience, we rename the object 0 in (25.5.8) as $1'$. Thus, in the n-fold coproduct

$$\mathscr{I}_2(\{p_1,\dots,p_n\}) \;=\; \mathscr{I}_2(\{p_1\}) \amalg \dots \amalg \mathscr{I}_2(\{p_n\})$$

the arrows corresponding to the i-th copy of (25.5.8) will be denoted as $\mu_i, \mu_i', a_i, a_i^*, \xi_i, \xi_i'$.

Then, the perversely thickened DG category $\mathscr{A}(D^2,\{p_1,\dots,p_n\})$ is given as the (homotopy) pushout[1] of the following diagram

$$\left[\mathscr{I}_2(\{p_1\}) \amalg \dots \amalg \mathscr{I}_2(\{p_n\}) \xleftarrow{(i_1',\dots,i_n')} k\langle\mu_1'^{\pm}\rangle \amalg \dots \amalg k\langle\mu_n'^{\pm}\rangle \xrightarrow{\mu_i' \mapsto T_i} k\langle T_1^{\pm},\dots,T_n^{\pm}\rangle\right] \tag{25.5.12}$$

A direct calculation of this pushout will then give the following presentation

$$\mathscr{A}(D^2,\{p_1,\dots,p_n\}) \;=\; \left[\begin{tikzcd} \boxed{k\langle\mu_1^{\pm}\rangle} \arrow[loop above, "\xi_1"] \arrow[dr, bend left, "a_1"] & \cdots & \boxed{k\langle\mu_n^{\pm}\rangle} \arrow[loop above, "\xi_n"] \arrow[dl, bend left, "a_n"] \\ & \boxed{k\langle T_1^{\pm},\dots,T_n^{\pm}\rangle} \arrow[loop below, "\xi_1',\dots,\xi_n'"] \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_n^*"] & \end{tikzcd}\right] \tag{25.5.13}$$

with differentials

$$d(\xi_i) = \mu_i - a_i^* a_i - 1 \qquad\qquad d(\xi_i') = T_i - a_i a_i^* - 1 \tag{25.5.14}$$

It will be shown in Proposition A.0.21 that the DG category (25.5.8) is acyclic in positive degree. Therefore, it is quasi-equivalent to its 0-th homology. We summarize this into the following

[1]It follows from Proposition A.0.8 and A.0.4 (see also Remark A.0.9) that the ordinary pushout of this diagram represents its homotopy pushout. In the remaining of Part IV of this thesis, when we write "(homotopy) pushout", we will implicitly mean that the ordinary pushout and the homotopy pushout coincide in that case.

Theorem 25.5.15. *The perversely thickened DG category $\mathscr{A}(D^2, \{p_1, \dots, p_n\})$ of the pair $(D^2, \{p_1, \dots, p_n\})$ is given by the k-category*

$$\mathscr{A}(D^2, \{p_1, \dots, p_n\}) \; = \; \left[\begin{tikzcd} \boxed{k\langle \mu_1^{\pm} \rangle} \arrow[dr, bend left, "a_1"] & \cdots & \boxed{k\langle \mu_n^{\pm} \rangle} \arrow[dl, bend left, "a_n"] \\ & \boxed{k\langle T_1^{\pm}, \dots, T_n^{\pm} \rangle} \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_n^*"] & \end{tikzcd} \right] \Big/ \begin{pmatrix} \mu_i = a_i^* a_i + 1 \\ T_i = a_i a_i^* + 1 \end{pmatrix} \tag{25.5.16}$$

Moreover, the perverse peripheral map

$$k\langle \mu_1^{\pm} \rangle \amalg \dots \amalg k\langle \mu_n^{\pm} \rangle \amalg k\langle Q^{\pm} \rangle \longrightarrow \mathscr{A}(D^2, \{p_1, \dots, p_n\}) \tag{25.5.17}$$

is given by

$$\mu_i \mapsto \mu_i \qquad\qquad Q_i \mapsto T_1 \dots T_n$$

This map has a canonical relative 2-Calabi-Yau structure.

We think of the perverse peripheral map (25.5.17) as sperifiying the non-central coefficients $\mu_1, \dots \mu_n, Q$ inside the k-category $\mathscr{A}(D^2, \{p_1, \dots, p_n\})$. To make these into central coefficients, we perform a boundary reduction.

Thus, we consider the map $k\langle \mu_i^{\pm} \rangle \hookrightarrow k\langle \mu_i^{\pm}, \nu_i \rangle$, where ν_i has degree 1, with differential $d(\nu_i) = \mu_i - t_i$, for an invertible element $t_i \in k^{\times}$ in the base commutative ring k. One can show that this map is quasi-equivalent to the peripheral map $\mathsf{AH}(S^1) \to \mathsf{AH}(D^2)$ associated to the disk D^2 with boundary S^1 (see (25.1.6)). Therefore, it has a canonical relative 2-Calabi-Yau structure.

Similarly, consider the map $k\langle Q^{\pm} \rangle \hookrightarrow k\langle Q^{\pm}, \tilde{Q} \rangle$, where $\tilde{Q}$ has degree 1, with differential $d(\tilde{Q}) = Q - q$, for an invertible element $q \in k^{\times}$. Then this map also has a canonical relative 2-Calabi-Yau structure.

We can use these maps to turn the non-central parameter (25.5.17) of the perversely thickened DG category into central ones. Formally, we take the homotopy pushout

$$\begin{tikzcd} k\langle\mu_1^\pm\rangle \amalg \dots \amalg k\langle\mu_n^\pm\rangle \amalg k\langle Q^\pm\rangle \arrow[r] \arrow[d] & \mathscr{A}(D^2,\{p_1,\dots,p_n\}) \arrow[d] \\ k\langle\mu_1^\pm,\nu_1\rangle \amalg \dots \amalg k\langle\mu_n^\pm,\nu_n\rangle \amalg k\langle Q^\pm,\tilde{Q}\rangle \arrow[r] & \mathscr{A}(D^2,\{p_1,\dots,p_n\})|_{(t_1,\dots,t_n,q)} \end{tikzcd}$$

The result is the DG category

$$\mathscr{A}(D^2,\{p_1,\dots,p_n\})|_{(t_1,\dots,t_n,q)} = \left[\begin{tikzcd} \bullet \arrow[loop above, "\nu_1"] \arrow[dr, shift left, "a_1"] & \cdots & \bullet \arrow[loop above, "\nu_n"] \arrow[dl, shift left, "a_n"] \\ & \bullet \arrow[ul, shift left, "a_1^*"] \arrow[ur, shift left, "a_n^*"] \arrow[loop below, "\tilde{Q}"] & \end{tikzcd} \right] \tag{25.5.18}$$

with differentials

$$d(\nu_i) = a_i^* a_i + 1 - t_i \qquad\qquad d(\tilde{Q}) = \Big(\prod_{i=1}^{n}(1 + a_i a_i^*)\Big) - q$$

This is called the *derived (multiplicative) preprojective algebra* in [17] because its 0-th homology

$$\mathrm{H}_0(\mathscr{A}(D^2,\{p_1,\dots,p_n\})|_{(t_1,\dots,t_n,q)}) = \left[\begin{tikzcd} 1 \arrow[dr, shift left, "a_1"] & \cdots & n \arrow[dl, shift left, "a_n"] \\ & 0 \arrow[ul, shift left, "a_1^*"] \arrow[ur, shift left, "a_n^*"] & \end{tikzcd} \right] \Bigg/ \begin{pmatrix} t_i = 1 + a_i^* a_i \\ q = \prod_{i=1}^n (1 + a_i a_i^*) \end{pmatrix} \tag{25.5.19}$$

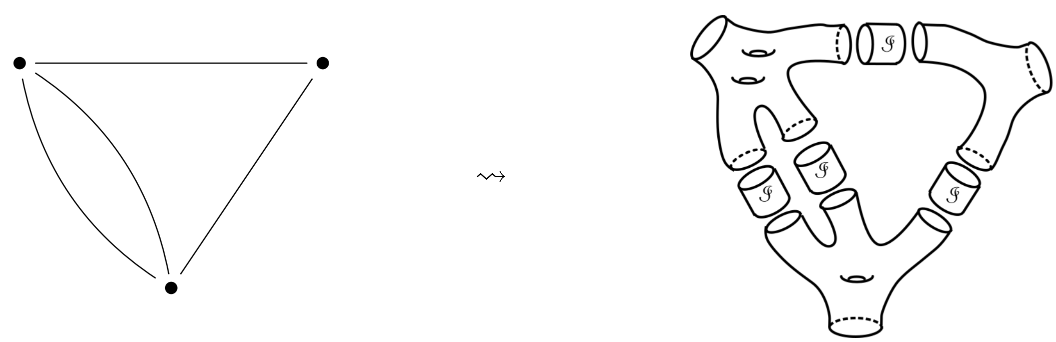
is essentially isomorphic to the multiplicative preprojective algebra defined and studied in [35]. Precisely, the *total algebra*, *i.e.*, the sum of all the Hom-complexes, of (25.5.19) is isomorphic to the multiplicative preprojective algebra as an algebra with idempotents.

Since the DG-category (25.5.18) is defined as a composition of 2-Calabi-Yau cospans, we have the following

Theorem 25.5.20. *The DG category* (25.5.18) *has a canonical* 2*-Calabi-Yau structure.*

Notice that a similar result has already appeared in [17, Theorem 1.7]. A conjectural relation with the present result is also briefly discussed in [17, Section 5.C].

Remark 25.5.21. In [17], R. Bezrukavnikov and M. Kapranov constructed higher genus generalizations of the (derived) multiplicative preprojective algebra. From the point of view of the present thesis, these DG categories can be obtained by gluing ther perverse neighborhood along different gluing patterns. Given a graph Γ, choose a genus g_v for each vertex $v \in \Gamma$. Suppose v is an n_v-valent vertex, then remove $n_v + 1$ disks from the genus g closed surface Σ_g. Then, glue in the perverse neighborhoods according to the shape of the graph Γ, as in the following example



This gives a DG category $\mathscr{A}(\Gamma; (g_v)_{v \in \mathrm{Vert}(\Gamma)})$ defined as a homotopy pushout. For each vertex, there will be a remaining S^1-boundary after the gluing. This induces a map from the Adams-Hilton model $\mathsf{AH}(S^1) \simeq k\langle \mu_v^{\pm} \rangle$ associated to each vertex v. The collection of these maps can be written as a map

$$\amalg_{v \in \mathrm{Vert}(\Gamma)} k\langle \mu_v^{\pm} \rangle \to \mathscr{A}(\Gamma; (g_v)_{v \in \mathrm{Vert}(\Gamma)})$$

which has a canonical relative 2-Calabi-Yau structure.

25.6 The link DG category

A link L in $\mathbb{R}^3$ can be equivalently thought of as a link L inside a closed ball N of large radius. Therefore, one can view this to be in the setting of Section 25.4. In this case, specifying a normal framing Φ is equivalent to specifying a framing of the link $L \subset \mathbb{R}^3$ in the classical sense.

We compute in this subsection the perversely thickened DG category for the pair $(\mathbb{R}^3, L)$. The result is the link DG category constructed in [11]. This DG category extends the Legendrian DG algebra [120, 121, 122, 54] of the unit conormal bundle $ST^*_L\mathbb{R}^3 \subset ST^*\mathbb{R}^3$.

As in the last subsection, we first compute the relative 3-Calabi-Yau completion of the directed cylinder

$$\mathcal{A} := k\langle v_1^{\pm}\rangle \amalg k\langle v_0^{\pm}\rangle \to \mathcal{A} \otimes \vec{I} \tag{25.6.1}$$

By the result of Section 24.1 (see (24.1.9)), one can take the semi-free resolution $\overrightarrow{\mathrm{Cyl}}_{\mathrm{BL}}(\mathcal{A})$ of the target $\mathcal{A} \otimes \vec{I}$ in (25.6.1). In other words, consider

$$\mathcal{A} := k\langle v_1^{\pm}\rangle \amalg k\langle v_0^{\pm}\rangle \xrightarrow{F} \left[\begin{tikzcd} \boxed{k\langle v_1^{\pm}\rangle} \arrow[r, bend left, "b"] \arrow[r, "a"'] & \boxed{k\langle v_0^{\pm}\rangle} \end{tikzcd} \right] =: \mathcal{B} \tag{25.6.2}$$

where the generators have degrees $|v_1| = |v_0| = |a| = 0$ and $|b| = 1$, with differential $d(b) = v_0 \cdot a - a \cdot v_1$.

To compute the relative 3-Calabi-Yau completion of (25.6.2), we first notice that the standarad resolution $\mathrm{Res}(\mathcal{A})$ of $\mathcal{A}$ is semi-free over the basis $\{E_1, E_0, sDv_1, sDv_0\}$ with differentials

$$\begin{aligned} d(sDv_1) &= v_1 \cdot E_1 - E_1 \cdot v_1 \\ d(sDv_0) &= v_0 \cdot E_0 - E_0 \cdot v_0 \end{aligned} \tag{25.6.3}$$

Similarly, $\mathrm{Res}(\mathcal{B})$ is semi-free over $\{E_1, E_0, sDv_1, sDv_2, sDa, sDb\}$ with differentials

$$
\begin{aligned}
d(sDv_1) &= v_1 \cdot E_1 - E_1 \cdot v_1 \\
d(sDv_0) &= v_0 \cdot E_0 - E_0 \cdot v_0 \\
d(sDa) &= a \cdot E_1 - E_0 \cdot a \\
d(sDb) &= b \cdot E_1 - E_0 \cdot b - sDv_0 \cdot a - v_0 \cdot sDa + sDa \cdot v_1 + a \cdot sDv_1
\end{aligned}
\tag{25.6.4}
$$

Therefore, the shifted cone

$$\Xi \;=\; \mathrm{cone}\left[\, \mathrm{Res}(\mathcal{B})^\vee \xrightarrow{\gamma_F^\vee} \mathrm{Res}(\mathcal{A})^\vee \,\right][1]$$

is semi-free over the basis

$$
\left\{ \begin{array}{c} sE^\vee_{1,\mathcal{A}}\,,\; sE^\vee_{0,\mathcal{A}}\,,\; s(sDv_{1,\mathcal{A}})^\vee\,,\; s(sDv_{0,\mathcal{A}})^\vee \\ s^2E^\vee_{1,\mathcal{B}}\,,\; s^2E^\vee_{0,\mathcal{B}}\,,\; s^2(sDv_{1,\mathcal{B}})^\vee\,,\; s^2(sDv_{0,\mathcal{B}})^\vee\,,\; s^2(sDa)^\vee\,,\; s^2(sDb)^\vee \end{array} \right\}
\tag{25.6.5}
$$

with differentials (see (22.1.5) and (22.1.6)) given by

$$
\begin{aligned}
d(s(sDv_{1,\mathcal{A}})^\vee) &= d(s(sDv_{0,\mathcal{A}})^\vee) = 0 \\
d(sE^\vee_{1,\mathcal{A}}) &= -v_1 \cdot s(sDv_{1,\mathcal{A}})^\vee + s(sDv_{1,\mathcal{A}})^\vee \cdot v_1 \\
d(sE^\vee_{0,\mathcal{A}}) &= -v_0 \cdot s(sDv_{0,\mathcal{A}})^\vee + s(sDv_{0,\mathcal{A}})^\vee \cdot v_0 \\
d(s^2(sDb)^\vee) &= 0 \\
d(s^2(sDa)^\vee) &= -v_1 \cdot s^2(sDb)^\vee + s^2(sDb)^\vee \cdot v_0 \\
d(s^2(sDv_{1,\mathcal{B}})^\vee) &= -s^2(sDb)^\vee \cdot a - s(sDv_{1,\mathcal{A}})^\vee \\
d(s^2(sDv_{0,\mathcal{B}})^\vee) &= a \cdot s^2(sDb)^\vee - s(sDv_{0,\mathcal{A}})^\vee \\
d(s^2E^\vee_{1,\mathcal{B}}) &= -s^2(sDv_{1,\mathcal{B}})^\vee \cdot v_1 + v_1 \cdot s^2(sDv_{1,\mathcal{B}})^\vee - s^2(sDa)^\vee \cdot a - s^2(sDb)^\vee \cdot b - sE^\vee_{1,\mathcal{A}} \\
d(s^2E^\vee_{0,\mathcal{B}}) &= -s^2(sDv_{0,\mathcal{B}})^\vee \cdot v_0 + v_0 \cdot s^2(sDv_{0,\mathcal{B}})^\vee + a \cdot s^2(sDa)^\vee + b \cdot s^2(sDb)^\vee + sE^\vee_{0,\mathcal{A}}
\end{aligned}
\tag{25.6.6}
$$

Rename the basis elements (25.6.5) as

$$\mu_1 := -s(sDv_{1,\mathcal{A}})^\vee \qquad \mu_0 := s(sDv_{0,\mathcal{A}})^\vee \tag{25.6.7}$$

$$t_1 := -sE_{1,\mathcal{A}}^\vee \qquad t_0 := sE_{0,\mathcal{A}}^\vee \tag{25.6.8}$$

$$a^* := s^2(sDb)^\vee \qquad b^* := s^2(sDa)^\vee \tag{25.6.9}$$

$$\xi_1 := s^2(sDv_{1,\mathcal{B}})^\vee \qquad \xi_0 := -s^2(sDv_{0,\mathcal{B}})^\vee \tag{25.6.10}$$

$$\eta_1 := -s^2E_{1,\mathcal{B}}^\vee \qquad \eta_0 := s^2E_{0,\mathcal{B}}^\vee \tag{25.6.11}$$

Then the relative 3-Calabi-Yau completion of (25.6.2) then has the presentation

$$\Pi_2(\mathcal{A}) = \left[\quad k\langle v_1^\pm, \mu_1, t_1\rangle \qquad \amalg \qquad k\langle v_0^\pm, \mu_0, t_0\rangle \quad \right] \tag{25.6.12}$$

and

$$\Pi_3(\mathcal{B},\mathcal{A}) = \left[\begin{tikzcd} \boxed{k\langle v_1^\pm, \mu_1, t_1\rangle} \arrow[loop above, "\xi_1"] \arrow[loop below, "\eta_1"] \arrow[r, bend left, "b"] \arrow[r, shift left, "a"] & \boxed{k\langle v_0^\pm, \mu_0, t_0\rangle} \arrow[loop above, "\xi_0"] \arrow[loop below, "\eta_0"] \arrow[l, shift left, "a^*"] \arrow[l, bend left, "b^*"] \end{tikzcd} \right] \tag{25.6.13}$$

The deformation parameter for the canonically deformed localized relative 3-Calabi-Yau completion (Definition 24.4.10) is given by the image of (25.4.2) under the map $\gamma_F : \mathrm{HC}_1^-(\mathcal{A}) \to \mathrm{HC}_1^-(\mathcal{B})$. Therefore, it is represented by

$$\gamma_F(\xi_{\mathcal{A}}) = sDv_{1,\mathcal{B}} - sDv_{0,\mathcal{B}} \quad \in \mathcal{B} \otimes_{\mathcal{B}^e} \mathrm{Res}(\mathcal{B})$$

Therefore, the deformed localized relative 2-Calabi-Yau completion of (25.6.2)

$$\Pi_2^{\mathrm{loc}}(\mathcal{A}) \to \Pi_3^{\mathrm{loc}}(\mathcal{B},\mathcal{A};\gamma_F(\xi_{\mathcal{A}})) \tag{25.6.14}$$

is given by the presentations (25.6.12) and (25.6.13), with the generators μ_1 and

μ_0 inverted, and with differentials given by

$$
\begin{aligned}
&d(v_1) = d(v_0) = d(a) = 0 \\
&d(\mu_1) = d(\mu_0) = d(a^*) = 0 \\
&d(b) = v_0 \cdot a - a \cdot v_1 \qquad\qquad d(b^*) = a^* v_0 - v_1 a^* \\
&d(t_1) = \mu_1 v_1 - v_1 \mu_1 \qquad\qquad d(t_0) = \mu_0 v_0 - v_0 \mu_0 \\
&d(\xi_1) = -a^* a + \mu_1 - 1 \qquad\qquad d(\xi_0) = -a a^* + \mu_0 - 1 \\
&d(\eta_1) = \xi_1 v_1 - v_1 \xi_1 + b^* a + a^* b - t_1 \\
&d(\eta_0) = \xi_0 v_0 - v_0 \xi_0 + a b^* + b a^* - t_0
\end{aligned}
\tag{25.6.15}
$$

Now, notice that $\Pi_2^{\mathrm{loc}}(\mathcal{A}) = k\langle v_1^\pm, \mu_1^\pm, t_1 \rangle \ \amalg \ k\langle v_0^\pm, \mu_0^\pm, t_0 \rangle$ is acyclic in positive degree, and is therefore quasi-equivalent to the alternative presentation

$$
\Pi_2^{\mathrm{loc}}(\mathcal{A}) = \left[\qquad k[v_1^\pm, \mu_1^\pm] \qquad\qquad\qquad k[v_0^\pm, \mu_0^\pm] \qquad \right]
\tag{25.6.16}
$$

Similarly, the DG category $\Pi_3^{\mathrm{loc}}(\mathcal{B}, \mathcal{A}; \gamma_F(\xi_{\mathcal{A}}))$ also has a simplified form

$$
\Pi_3^{\mathrm{loc}}(\mathcal{B}, \mathcal{A}; \gamma_F(\xi_{\mathcal{A}})) \ = \ \left[\begin{tikzcd} \boxed{k[v_1^\pm, \mu_1^\pm]} \arrow[loop above, "\xi_1"] \arrow[loop below, "\eta_1"] \arrow[r, bend left, "b"] \arrow[r, shift left, "a"] & \boxed{k[v_0^\pm, \mu_0^\pm]} \arrow[loop above, "\xi_0"] \arrow[loop below, "\eta_0"] \arrow[l, shift left, "a^*"] \arrow[l, bend left, "b^*"] \end{tikzcd} \right]
\tag{25.6.17}
$$

This gives a presentation of the 3-dimensional perverse neighborhood

$$
\mathscr{I}_3(S^1) \ = \ \Pi_3^{\mathrm{loc}}(\mathcal{B}, \mathcal{A}; \gamma_F(\xi_{\mathcal{A}}))
\tag{25.6.18}
$$

Now, we glue this perverse neighborhood with the Adams-Hilton model $\mathsf{AH}(X)$ of the link complement $X = \mathbb{R}^3 \setminus \nu(L)$. The normal framing Φ specifies a map $\mathsf{AH}(\Phi) : \mathsf{AH}(S^1 \times S^1) \to \mathsf{AH}(X)$. Since the 2-torus $S^1 \times S^1$ is aspherical, its Adams-Hilton model is simply given by its group algebra. *i.e.*, we have $\mathsf{AH}(S^1 \times S^1) \simeq k[\lambda^\pm, \mu^\pm]$, where μ is the meridian, and λ is the longitude determined by the normal framing Φ.

The localized 2-Calabi-Yau completion of $\mathcal{A}$ can then be identified with the disjoint union $\mathsf{AH}(S^1 \times S^1) \amalg \mathsf{AH}(S^1 \times S^1)'$, where we identify the generators (v_1, μ_1, v_0, μ_0) in (25.6.16) with the generators $(\lambda_1^{-1}, \mu_1, \lambda_1'^{-1}, \mu_1')$ in the disjoint union $\mathsf{AH}(S^1 \times S^1) \amalg \mathsf{AH}(S^1 \times S^1)'$. For convenience of notation, rename the object 0 in (25.6.17) as $1'$, and rename the generators accordingly.

Thus, the perversely thickened DG category $\mathscr{A}(\mathbb{R}^3, L; \Phi)$ is given by the (homotopy) colimit of the following diagram

$$\left[\quad \mathscr{I}_3(S^1) \amalg \overset{r}{\cdots} \amalg \mathscr{I}_3(S^1) \xleftarrow{(i_1', \ldots, i_r')} \mathsf{AH}(S^1 \times S^1) \amalg \overset{r}{\cdots} \amalg \mathsf{AH}(S^1 \times S^1) \longrightarrow \mathsf{AH}(X) \quad \right] \tag{25.6.19}$$

where r is the number of components of the link L.

The result of this calculation is summarized in the following

Theorem 25.6.20. *The perversely thickened DG category $\mathscr{A}(\mathbb{R}^3, L; \Phi)$ associated to a framed link $L \subset \mathbb{R}^3$ has the presentation*

$$\mathscr{A}(\mathbb{R}^3, L; \Phi) = \left[\begin{tikzcd}
\boxed{k[\lambda_1^{\pm}, \mu_1^{\pm}]} \arrow[loop above, "\xi_1"] \arrow[loop below, "\eta_1"] \arrow[dr, bend left, "b_1"] \arrow[dr, bend left=10, "a_1"] & \cdots & \boxed{k[\lambda_r^{\pm}, \mu_r^{\pm}]} \arrow[loop above, "\xi_r"] \arrow[loop below, "\eta_r"] \arrow[dl, bend right=10, "a_r"] \arrow[dl, bend left, "b_r"] \\
& \boxed{\mathsf{AH}(X)} \arrow[loop above, "\xi_1', \ldots, \xi_r'"] \arrow[loop below, "\eta_1' \ldots, \eta_r'"] \arrow[ul, bend left=10, "a_1^*"] \arrow[ul, bend left=30, "b_1^*"] \arrow[ur, bend right=10, "a_r^*"] \arrow[ur, bend left, "b_r^*"] &
\end{tikzcd} \right] \tag{25.6.21}$$

with differentials given by

$$\begin{aligned}
&d(\lambda_i) = d(\mu_i) = d(a_i) = d(a_i^*) = 0 \\
&d(b_i) = \phi_i(\lambda_i)^{-1} \cdot a_i - a_i \cdot \lambda_i^{-1} \qquad\qquad d(b_i^*) = a_i^* \lambda_i^{-1} - \phi_i(\lambda_i)^{-1} a_i^* \\
&d(\xi_i) = -a_i^* a_i + \mu_i - 1 \qquad\qquad d(\xi_i') = -a_i a_i^* + \phi_i(\mu_i) - 1 \\
&d(\eta_i) = \xi_i \lambda_i^{-1} - \lambda_i^{-1} \xi_i + b_i^* a_i + a_i^* b_i \\
&d(\eta_i') = \xi_i' \phi_i(\lambda_i)^{-1} - \phi_i(\lambda_i)^{-1} \xi_i' + a_i b_i^* + b_i a_i^*
\end{aligned} \tag{25.6.22}$$

where $\phi_i : k[\mu_i, \lambda_i] \to \mathsf{AH}(X)$ is the map of Adams-Hilton models for the inclusion of the i-th torus boundary of X.

The inclusion $S^2 \to X$ of a sphere of large radius into $X = \mathbb{R}^3 \setminus \nu(L)$ induces a canonical map $k\langle\tau\rangle \overset{(25.1.5)}{\cong} \mathsf{AH}(S^2) \to \mathsf{AH}(X)$ where τ has degree 1. This gives a canonical map

$$k[\lambda_1^{\pm}, \mu_1^{\pm}] \amalg \overset{r}{\cdots} \amalg k[\lambda_r^{\pm}, \mu_r^{\pm}] \amalg k\langle\tau\rangle \to \mathscr{A}(\mathbb{R}^3, L; \Phi) \tag{25.6.23}$$

Theorem 25.6.24. *The map* (25.6.23) *is the perverse peripheral map (Definition 25.4.6). This map has a canonical relative* 3*-Calabi-Yau structure.*

It is shown in [11] that the DG category (25.6.21) is quasi-equivalent to the link DG category constructed in [11]. In particular, if we denote by $\mathscr{A}(\mathbb{R}^3, L; \Phi)|_{\{1,\ldots,r\}\to\{1\}}$ the DG category obtained by collapsing the objects $\{1, \ldots, r\}$ in (25.6.21) to a single object $\{1\}$, then we have

Theorem 25.6.25 ([11])**.** *The endomorphism DG algebra of the DG category* $\mathscr{A}(\mathbb{R}^3, L; \Phi)|_{\{1,\ldots,r\}\to\{1\}}$ *at the object* 1 *is quasi-isomorphic to the Legendrian DG algebra [120, 121, 122, 54] of the unit conormal bundle* $ST^*_L\mathbb{R}^3 \subset ST^*\mathbb{R}^3$.

25.7 Relations with perverse sheaves

In both of the examples in the last two subsections, the category of finite dimensional modules over the 0-th homology of the perversely thickened DG category of the pair (N, M) is equivalent to the category $\mathsf{Perv}(N, M)$ of perverse sheaves on N constructible with respect to the stratification $\{M, N \setminus M\}$. The case for $(N, M) = (D^2, \{p_1, \ldots, p_n\})$ was shown in [66], while the case for

$(N, M) = (\mathbb{R}^3, L)$ was shown in [11] (see also [10]). We show in this subsection that this result is true in general.

The first step is to compute the 0-th homology of the perverse neighborhood. The result is given by the following proposition, whose proof is a formalization of the calculations in the last two subsections.

Proposition 25.7.1. *The n-dimensional perverse neighborhood $\mathscr{I}_n(M)$ of any connected closed oriented manifold M of dimension $n-2$ has 0-th homology given by*

$$\mathrm{H}_0(\,\mathscr{I}_n(M)\,) \;=\; \left[\; \begin{tikzcd} \boxed{k[\pi_1(M)] \otimes k\langle \mu^{\pm}\rangle} \arrow[r, shift left, "a"] & \boxed{k[\pi_1(M)]' \otimes k\langle \mu'^{\pm}\rangle} \arrow[l, shift left, "a^*"] \end{tikzcd} \;\right] \Big/ \; I$$

where I is the ideal generated by

$$\mu = a^*a + 1$$

$$\mu' = aa^* + 1$$

$$\lambda' a = a\lambda \quad \textit{for all } \lambda \in \pi_1(M)$$

$$\lambda a^* = a^* \lambda' \quad \textit{for all } \lambda \in \pi_1(M)$$

Proof. Choose a reduced CW cell structure on M, then by the discussion in Section 25.1, see (25.1.6), one can choose an Adams-Hilton DG algebra model $A = \mathsf{AH}(M)$ of the form

$$A \;=\; k\langle x_1^{\pm}, x_2^{\pm}, \ldots, z_1, \ldots z_p\rangle$$

where we have denoted the top degree generators as $z_1, \ldots, z_p$. Thus they have degree $|z_i| = n-3$.

The fundamental class on the Adams-Hilton model A gives rise to a quasi-isomorphism

$$\hat{\xi} \;:\; \mathrm{Res}(A)^{\vee}[n-2] \;\xrightarrow{\sim}\; A \tag{25.7.2}$$

We need to compute the localized relative n-Calabi-Yau completion of the DG functor

$$\mathcal{A} := A \amalg A' \to A \otimes \vec{I}$$

By (25.7.2), one can take the following model for the directed cylinder $A \otimes \vec{I}$.

$$\begin{aligned} \mathcal{B} &= T_{A \amalg A}\left(\operatorname{Res}(A)^{\vee}_{21}[n-2] \right) \\ &= \left[\begin{tikzcd} \boxed{A} \arrow[r, "{[E_0^\vee],[x_1^\vee],\ldots,[z_p^\vee]}"] & \boxed{A'} \end{tikzcd} \right] \end{aligned}$$

where the generators $[E_0^\vee], [x_1^\vee], \ldots, [z_p^\vee]$ are the ones correspondings to the basis element $[E_0^\vee] := s^{n-2}E_0^\vee$, $[x_1^\vee] := s^{n-2}(sDx_1)^\vee$, etc, in $\operatorname{Res}(A)^\vee$.

The relative n-Calabi-Yau completion of the inclusoin $F : \mathcal{A} \to \mathcal{B}$ is therefore given by

$$\Pi_n(\mathcal{B}, \mathcal{A}) = \left[\begin{tikzcd} \boxed{A} \arrow[loop above, "S_3"] \arrow[loop below, "S_4"'] \arrow[r, bend left, "S_1"] & \boxed{A'} \arrow[loop above, "S_3'"] \arrow[loop below, "S_4'"'] \arrow[l, bend left, "S_2"] \end{tikzcd} \right] \tag{25.7.3}$$

where $S_1, S_2, S_3, S_4, S_3', S_4'$ are the sets of generators

$$\begin{aligned} S_1 &= \{ [E_0^\vee], [x_1^\vee], \ldots, [z_p^\vee] \} \\ S_2 &= \{ s^{n-1}(sD[E_0^\vee])^\vee, s^{n-1}(sD[x_1^\vee])^\vee, \ldots, s^{n-1}(sD[z_p^\vee])^\vee \} \\ S_3 &= \{ s^{n-2}(E_{0,\mathcal{A}})^\vee, s^{n-2}(sDx_{1,\mathcal{A}})^\vee, \ldots, s^{n-2}(sDz_{p,\mathcal{A}})^\vee \} \\ S_4 &= \{ s^{n-1}(E_{0,\mathcal{B}})^\vee, s^{n-1}(sDx_{1,\mathcal{B}})^\vee, \ldots, s^{n-1}(sDz_{p,\mathcal{B}})^\vee \} \\ S_3' &= \{ s^{n-2}(E_{0,\mathcal{A}}')^\vee, s^{n-2}(sDx_{1,\mathcal{A}}')^\vee, \ldots, s^{n-2}(sDz_{p,\mathcal{A}}')^\vee \} \\ S_4' &= \{ s^{n-1}(E_{0,\mathcal{B}}')^\vee, s^{n-1}(sDx_{1,\mathcal{B}}')^\vee, \ldots, s^{n-1}(sDz_{p,\mathcal{B}}')^\vee \} \end{aligned} \tag{25.7.4}$$

Under the canonical deformation parameter $\eta_{\mathcal{B}} = \gamma_F(\xi_{\mathcal{A}})$, the differentials of the degree 1 generators $s^{n-1}(sDz_{i,\mathcal{B}})^\vee \in S_4$ and $s^{n-1}(sDz'_{i,\mathcal{B}})^\vee \in S'_4$ are given by

$$\begin{aligned} d(s^{n-1}(sDz_{i,\mathcal{B}})^\vee) &= (-1)^{n-2}\big[s^{n-2}(sDz_{i,\mathcal{A}})^\vee + s^{n-1}(sD[E_0^\vee])^\vee \cdot [z_i^\vee] + \hat{\xi}((sDz_i)^\vee)\big] \\ d(s^{n-1}(sDz'_{i,\mathcal{B}})^\vee) &= (-1)^{n-2}\big[s^{n-2}(sDz'_{i,\mathcal{A}})^\vee - [z_i^\vee] \cdot s^{n-1}(sD[E_0^\vee])^\vee - \hat{\xi}((sDz_i)^\vee)\big] \end{aligned} \tag{25.7.5}$$

In order to compute the 0-th homology, it suffices to consider only the generators of degrees 0 and 1. In fact, we will perform a simplification of the DG category (25.7.3), so as to further reduce the number of such low degree generators.

Consider the following DG subcategory of (25.7.3)

$$\mathcal{E} = \left[\begin{tikzcd} \boxed{A} \arrow[loop above, "S_3"] \arrow[rr, bend left=10, "S_1"] & & \boxed{A'} \arrow[loop above, "S'_3"] \end{tikzcd} \right] \tag{25.7.6}$$

The subcategory $A\langle S_3\rangle$ is simply the $(n-1)$-Calabi-Yau completion of A, and is therefore quasi-isomorphic to $A \otimes k[\mu]$. The same is true for $A'\langle S'_3\rangle$. This gives a quasi-equivalence $G : A \amalg A' \to A \otimes k[\mu] \amalg A' \otimes k[\mu']$.

Combining with the quasi-isomorphism (25.7.2), one has a sequence of quasi-equivalences

$$\begin{aligned} \mathcal{E} &\to \left[\begin{tikzcd} \boxed{A \otimes k[\mu]} \arrow[rr, bend left=5, "S_1"] & & \boxed{A' \otimes k[\mu']} \end{tikzcd} \right] \\ &= T_{A\otimes k[\mu] \amalg A' \otimes k[\mu']}(\, G_!(\mathrm{Res}(A)_{21}^\vee[n-2])\,) \\ &\xrightarrow{\hat{\xi}} T_{A\otimes k[\mu] \amalg A' \otimes k[\mu']}(\, G_!(A_{21})\,) \end{aligned} \tag{25.7.7}$$

We denote by φ the composition of the quasi-equivalences in (25.7.7).

Notice that the bimodule $G_!(A_{21})$ is generated by a single arrow, which we denote as a, modulo the relations $\lambda' a = a\lambda$ for all $\lambda \in A$. Therefore, we may rewrite the target of (25.7.7) as

$$T_{A\otimes k[\mu] \amalg A' \otimes k[\mu']}(\, G_!(A_{21}) = \left[\begin{tikzcd} \boxed{A \otimes k[\mu]} \arrow[r, "a"] & \boxed{A' \otimes k[\mu']} \end{tikzcd} \right] \Big/ (\lambda' a = a\lambda)$$

Under this identification, the map φ can be rewritten as

$$\varphi\,:\,\mathcal{E}\;=\;\left[\begin{tikzcd} \boxed{A} \arrow[loop above, "S_3"] \arrow[rr, "S_1"] & & \boxed{A'} \arrow[loop above, "S_3'"] \end{tikzcd}\right] \xrightarrow{\sim} \left[\begin{tikzcd} \boxed{A\otimes k[\mu]} \arrow[r, "a"] & \boxed{A'\otimes k[\mu']} \end{tikzcd}\right]\Big/ (\lambda' a = a\lambda)$$

This image of this map on the degree 0 generators in S_1, S_3, S_3' are given by

$$\begin{aligned}
\varphi([z_i^\vee]) \;&=\; \hat{\xi}((sDz_i)^\vee)a \\
\varphi(s^{n-2}(sDz_{i,\mathcal{A}})^\vee) \;&=\; -\hat{\xi}((sDz_i)^\vee)\mu \\
\varphi(s^{n-2}(sDz_{i,\mathcal{A}}')^\vee) \;&=\; \hat{\xi}((sDz_i)^\vee)\mu'
\end{aligned} \tag{25.7.8}$$

Now, form the (homotopy) pushout

$$\Pi' \;:=\; \operatorname{hocolim}\Big[\;\; T_{A\otimes k[\mu]\,\amalg\, A'\otimes k[\mu']}(\,G_!(A_{21})\,) \xleftarrow{\varphi} \mathcal{E} \hookrightarrow \Pi_n(\mathcal{B},\mathcal{A};\eta) \;\;\Big]$$

Since φ is a quasi-equivalence, so is its homotopy pushout φ' : $\Pi_n(\mathcal{B},\mathcal{A};\eta) \to \Pi'$. This homotopy pushout can be presented as

$$\Pi' \;=\; \left[\begin{tikzcd} \boxed{A\otimes k[\mu]} \arrow[loop below, "S_4"] \arrow[r, shift left, "a"] & \boxed{A'\otimes k[\mu']} \arrow[loop below, "S_4'"] \arrow[l, shift left, "S_2"] \end{tikzcd}\right]\Big/ (\lambda' a = a\lambda)$$

Because of (25.7.8), the formulas (25.7.5) takes the following form in the pushout Π'

$$\begin{aligned}
d(s^{n-1}(sDz_{i,\mathcal{B}})^\vee) &= (-1)^{n-2}\big[-\hat{\xi}((sDz_i)^\vee)\mu + s^{n-1}(sD[E_0^\vee])^\vee\cdot\hat{\xi}((sDz_i)^\vee)a + \hat{\xi}((sDz_i)^\vee)\big] \\
&= (-1)^{n-2}\big[-\mu + s^{n-1}(sD[E_0^\vee])^\vee\cdot a + 1\big]\cdot\hat{\xi}((sDz_i)^\vee) \\
d(s^{n-1}(sDz_{i,\mathcal{B}}')^\vee) &= (-1)^{n-2}\big[\hat{\xi}((sDz_i')^\vee)\mu' - \hat{\xi}((sDz_i)^\vee)a\cdot s^{n-1}(sD[E_0^\vee])^\vee - \hat{\xi}((sDz_i)^\vee)\big] \\
&= (-1)^{n-2}\hat{\xi}((sDz_i)^\vee)\cdot\big[\mu' - a\cdot s^{n-1}(sD[E_0^\vee])^\vee\cdot -1\big]
\end{aligned} \tag{25.7.9}$$

In the generating set S_2, the only degree 0 generator is $s^{n-1}(sD[E_0^\vee])^\vee$, which we denote as a^*. Its degree one generators are given by $[x_i^\vee]^\vee$, with differentials

$d([x_i^\vee]^\vee) = x_i a^* - a^* x_i'$. Therefore, at the level of 0-th homology, these differentials impose relations $\lambda a^* = a^* \lambda'$ for all $\lambda \in H_0(A)$.

The lowest degree elements in the generating sets S_4 and S_4' are the elements $s^{n-1}(sDz_{i,\mathcal{B}})^\vee \in S_4$ and $s^{n-1}(sDz'_{i,\mathcal{B}})^\vee \in S_4'$, which have degree 1. Their differentials are given by (25.7.9). Notice that, since $\hat{\xi} : \mathrm{Res}(A)^\vee[n-2] \to A$ is a quasi-isomorphism, an A_0-bilinear combination of the elements $\hat{\xi}((sDz_i)^\vee) \in A_0$ is homologous to the identity $1 \in A$. Therefore, at the level of 0-th homology, the differentials (25.7.9), for $1 \leq i \leq p$, impose relations that are equivalent to the relations

$$-\mu + a^* a + 1 = 0 \qquad \text{and} \qquad \mu' - a^* a - 1 = 0$$

In other words, we have

$$\mathrm{H}_0(\Pi(\mathcal{B}, \mathcal{A}; \eta)) \cong \mathrm{H}_0(\Pi') \;=\; \mathrm{H}_0 \left[\begin{tikzcd} \boxed{A \otimes k[\mu]} \arrow[loop below, "S_4"] \arrow[r, bend left, "a"] & \boxed{A' \otimes k[\mu']} \arrow[loop below, "S_4'"] \arrow[l, bend left, "S_2"] \end{tikzcd} \right] \Big/ (\lambda' a = a\lambda)$$

$$\cong \left[\begin{tikzcd} \boxed{A \otimes k[\mu]} \arrow[r, bend left, "a"] & \boxed{A' \otimes k[\mu']} \arrow[l, bend left, "a^*"] \end{tikzcd} \right] \Big/ \begin{pmatrix} \mu = a^* a + 1 \\ \mu' = a a^* + 1 \\ \lambda' a = a \lambda \\ \lambda a^* = a^* \lambda' \end{pmatrix}$$

The proof is then finished by localizing the elemnts μ and μ^{-1}. *i.e.*,

$$\mathrm{H}_0(\mathscr{I}_n(M)) \;=\; \mathrm{H}_0(\Pi^{\mathrm{loc}}(\mathcal{B}, \mathcal{A}; \eta)) \;=\; \mathrm{H}_0(\Pi(\mathcal{B}, \mathcal{A}; \eta))[\mu^{-1}, \mu'^{-1}]$$

□

An immediate consequence of this calculation is a description of the 0-th homology of the perversely deformed DG category. Suppose that the submanifold M has r components $M = M_1 \cup \ldots \cup M_r$, Choose a path γ_i from the base point of

$S_{M_i}N \subset \partial X$ to the basepoint of $X = N \setminus \nu(M)$. Together with the normal framing Φ, this induces a map $\phi_i : \pi_1(M_i \times S^1) \to \pi_1(X)$ of fundamental groups. Using this map, one has the following description of $\mathrm{H}_0(\mathscr{A}(N, M; \Phi))$.

Theorem 25.7.10. *The* 0*-th homology of the perversely deformed DG category* $\mathscr{A}(N, M; \Phi)$ *is given by*

$$\mathrm{H}_0(\mathscr{A}(N,M;\Phi)) = \left[\begin{tikzcd} \boxed{k[\pi_1(M_1)] \otimes k\langle \mu_1^{\pm} \rangle} \arrow[dr, bend left, "a_1"] & \cdots & \boxed{k[\pi_1(M_r)] \otimes k\langle \mu_r^{\pm} \rangle} \arrow[dl, bend left, "a_r"] \\ & \boxed{k[\pi_1(X)]} \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_r^*"] & \end{tikzcd} \right] \Big/ I$$

where I *is the ideal generated by*

$$\begin{aligned} \mu_i &= a_i^* a_i + 1 \\ \phi_i(\mu_i) &= a_i a_i^* + 1 \\ \phi_i(\lambda_i) a_i &= a_i \lambda_i \quad \textit{for all } \lambda_i \in \pi_1(M_i) \\ \phi_i(\lambda_i) a_i^* &= a_i^* \phi(\lambda_i) \quad \textit{for all } \lambda_i \in \pi_1(M_i) \end{aligned}$$

Readers who are familiar with the description of perverse sheaves in terms of the variation and canonical maps between nearby cycle and vanishing cycle functors will find that the formulas $\mu_i = a_i^* a_i + 1$ appearing in Theorem 25.7.10 resembles the formula $\mu = \text{var} \circ \text{can} + 1$. A precise relation can be obtained by appealing to the topological approach [118] to the nearby cycle and vanishing cycle functors, and is summarized in the following theorem. The convention for the perversity function in this theorem will be discussed after the theorem.

Theorem 25.7.11. *If* k *is a field, then the category* $\mathtt{Mod}^{\mathrm{fd}}\big(\mathrm{H}_0(\mathscr{A}(N, M; \Phi))\big)$ *of finite dimensional (left) modules of the* 0*-th homology of the perversely thickened DG category is equivalent to the category* $\mathsf{Perv}(N, M)$ *of perverse sheaves on* N *with singularities at most along* M*.*

This theorem can be proved by combining Theorem 25.7.10 with the results in [118]. Recall that [118] gives a gluing description of perverse sheaves for very general stratifications. We give a brief summary of this description in our simple case of the stratification $N = M \cup N\backslash M$. We will specify our degree conventions in this summary.

We will work with cohomological grading in the remaining of this subsection. Denote by $\mathcal{D}^b_c(N)$ the derived category of sheaves on N with bounded cohomology sheaves that are locally constant of finite rank. Given a perversity function $p : \mathbb{N} \to \mathbb{Z}$, we will write $p(Y) = p(\dim(Y))$ for any manifold Y. Then the category $\mathsf{Perv}^p(N, M)$ of p-perverse sheaves on N with singularities at most along M is defined (see [90]) to be the full subcategory $\mathsf{Perv}^p(N, M) \subset \mathcal{D}^b_c(N)$ of complexes $\mathcal{F} \in \mathcal{D}^b_c(N)$ of sheaves satisfying

$$i^{-1}(\mathcal{F}) \in \mathcal{D}^{\leq p(M)}_c(N) \qquad \text{and} \qquad i^!(\mathcal{F}) \in \mathcal{D}^{\geq p(M)}_c(N)$$

$$j^{-1}(\mathcal{F}) \in \mathcal{D}^{\leq p(N\backslash M)}_c(N) \qquad \text{and} \qquad j^!(\mathcal{F}) \in \mathcal{D}^{\geq p(N\backslash M)}_c(N)$$

where $i : M \hookrightarrow N$ and $j : N\backslash M \hookrightarrow M$ are the inclusion maps of the strata. Notice that, for the open inclusion j, we have $j^{-1} = j^!$. Therefore, the restriction condition on the strata $N\backslash M$ requires the complex $j^{-1}\mathcal{F}$ to be quasi-isomorphic to a locally constant sheaf concentrated in degree $p(N)$.

We will consider 'middle perversity'. This corresponds to taking $p(l) = -l/2$. Thus, for our stratification, this means $p(N \setminus M) = -n/2$ and $p(M) = -n/2 + 1$. However, we will deal with cases when N is odd dimensional. For this reason, we perform an overall shift, and take $p(N \setminus M) = 0$ and $p(M) = 1$. This guarantees that the perversity function is well-defined in all cases. Moreover, since an overall shift in the perversity function does not affect the category of perverse sheaves, this convention gives rise to the usual perverse sheaves of middle perversity in the case when N has even dimension. We will still call this the 'middle perversity', and will always work with this perversity function. The corresponding category of perverse sheaves will be simply denoted as $\mathsf{Perv}(N, M)$.

In [118], MacPherson and Vilonen gave a simple description of the category of perverse sheaves in terms of a gluing category. In the setting of [118], a space X has a stratification $\mathscr{S}$. Suppose that $U \in \mathscr{S}$ is an open strata with complement S, then the gluing category $\mathcal{M}(U, S; T)$prescirbes the data required to glue a local system on U with a perverse sheaf on S. In our case, the stratification is of the form $(M, N \setminus M)$. Therefore, we may take $S = M$ and $U = N \setminus M$. Since $S = M$ is itself a stratum, the gluing category $\mathcal{M}(U, S; T)$ specifies gluing data between local systems on U and S.

The crucial ingredient in the definition of this gluing category is a functor

$$T \,:\, \mathtt{Loc}(U) \to \mathrm{Mor}(\,\mathtt{Loc}(S)\,) \qquad\qquad A \mapsto \big[\,F(A) \xrightarrow{T} G(A)\,\big]. \tag{25.7.12}$$

which associates a map of local systems on S to every local system A on U.

Given this functor, one can define the category $\mathcal{M}(U, S; T)$ whose objects consist of pairs (A, B), where A is a local system on U, and B is a local system on S, together with two maps $a^* : F(A) \to B$ and $a : B \to G(A)$ that factorizes the map $T_A : F(A) \to G(A)$. *i.e.*, we have $T = aa^*$. Morphisms in the category $\mathcal{M}(U, S; T)$ are pairs consisting of a map $A_1 \to A_2$ in $\mathtt{Loc}(U)$ and a map $B_1 \to B_2$ in $\mathtt{Loc}(S)$ make the obvious diagrams commute.

To describe the functor, we first consider the following situation. Let $\pi : S^1 \to *$ be the unique map from the circle to a point, and consider the derived global section functor

$$\boldsymbol{R}\pi_* : \mathcal{D}_c^b(S^1) \to \mathcal{D}_c^b(*)$$

Notice that the category of local systems on S^1 is equivalent to the category of representations of $k[\pi_1(S^1)] \cong k[\mu^\pm]$. The equivalence $\mathtt{Loc}(S^1) \simeq \mathtt{Mod}^{\mathrm{fd}}(k[\mu^\pm])$ is obtained by taking the stalk $\mathcal{G}_x$ of a local system $\mathcal{G} \in \mathtt{Loc}(S^1)$ at the basepoint x of S^1. Moreover, if we take $\mathcal{G}$ to be a local system corresponding to the module V over $k[\mu^\pm]$, then its derived global section chain complex is simply given by (see, e.g. [135, Section II.6])

$$\boldsymbol{R}\pi_*(V) \simeq \boldsymbol{R}\underline{\mathrm{Hom}}_{k[\mu^\pm]}(k, V) \simeq [\,0 \to V \xrightarrow{\mu-1} V \to 0\,] \tag{25.7.13}$$

which is a complex concentrated in degree 0 and 1.

This complex has an alternative description. Let $K := \{x\} \subset S^1 =: L$ be the basepoint of L, and denote by κ and γ the inclusion maps $\kappa : K \hookrightarrow L$ and $\gamma : L \setminus K \to L$. Then, for any complex of sheaves $\mathcal{G}$ on L, there is a canonical triangle

$$\cdots \to \gamma_!\gamma^{-1}\mathcal{G} \to \mathcal{G} \to \kappa_*\kappa^{-1}\mathcal{G} \to \gamma_!\gamma^{-1}\mathcal{G}[1] \to \cdots$$

Applying the (exact) functor $\boldsymbol{R}\pi_*$ gives the following triangle in $\mathcal{D}_c^b(*)$

$$\cdots \to \boldsymbol{R}\pi_*\gamma_!\gamma^{-1}\mathcal{G} \to \boldsymbol{R}\pi_*\mathcal{G} \to \boldsymbol{R}\pi_*\kappa_*\kappa^{-1}\mathcal{G} \xrightarrow{\delta} \boldsymbol{R}\pi_*\gamma_!\gamma^{-1}\mathcal{G}[1] \to \cdots \tag{25.7.14}$$

If $\mathcal{G}$ is the local system corresponding to a module V over $k[\mu^\pm]$, then the map $\boldsymbol{R}\pi_*\mathcal{G} \to \boldsymbol{R}\pi_*\kappa_*\kappa^{-1}\mathcal{G}$ in (25.7.14) corresponds under (25.7.13) to the map

$$[\,0 \to V \xrightarrow{\mu-1} V \to 0\,] \to V$$

to its degree 0 part.

By uniqueness of cone, the complex $\boldsymbol{R}\pi_*\gamma_!\gamma^{-1}\mathcal{G}[1]$ is quasi-isomorphic to the vector space V concentrated in degree 0, and the map $\boldsymbol{R}\pi_*\kappa_*\kappa^{-1}\mathcal{G} \xrightarrow{\delta}$

$\boldsymbol{R}\pi_*\gamma_!\gamma^{-1}\mathcal{G}[1]$ can be identified with the map $\mu - 1 : V \to V$. This gives a description of the functor

$$T_0 : \mathtt{Loc}(S^1) \to \mathrm{Mor}(\mathtt{Loc}(\{*\})), \qquad \mathcal{G} \mapsto \left[\mathrm{H}^0\left(\boldsymbol{R}\pi_*\kappa_*\kappa^{-1}\mathcal{G}\right) \xrightarrow{\delta} \mathrm{H}^0\left(\boldsymbol{R}\pi_*\gamma_!\gamma^{-1}\mathcal{G}[1]\right) \right]$$

This functor, together with its linear algebraic description, extends to the case when the point $\{*\}$ is replaced by a manifold S, and the circle S^1 is replaced by a trivial S^1-bundle $\pi : L \to S$ over S. Suppose this bundle has a section $s : S \to L$ whose image is denoted by K. Denote by κ and γ the inclusion maps $\kappa : K \hookrightarrow L$ and $\gamma : L \setminus K \to L$. Since L is trivialized by the section s, the category of local systems on L can again be identified with the category $\mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(S)] \otimes k[\mu^{\pm}])$. Under this equivalence, the derived pushforward of a local system on L under $\pi : L \to S$ is given by

$$\boldsymbol{R}\pi_*(V) \simeq \boldsymbol{R}\underline{\mathrm{Hom}}_{k[\pi_1(S)]\otimes k[\mu^{\pm}]}(k[\pi_1(S)], V) \simeq [0 \to V \xrightarrow{\mu-1} V \to 0]$$

The same argument then allows us to give a linear algebraic description of the following functor

$$T_0 : \mathtt{Loc}(L) \to \mathrm{Mor}(\mathtt{Loc}(S)), \qquad \mathcal{G} \mapsto \left[\mathrm{H}^0\left(\boldsymbol{R}\pi_*\kappa_*\kappa^{-1}\mathcal{G}\right) \xrightarrow{\delta} \mathrm{H}^0\left(\boldsymbol{R}\pi_*\gamma_!\gamma^{-1}\mathcal{G}[1]\right) \right] \tag{25.7.15}$$

This can be summarized as

Proposition 25.7.16. *For any local system $\mathcal{G} \in \mathtt{Loc}(L)$, both the complexes $\boldsymbol{R}\pi_*\kappa_*\kappa^{-1}\mathcal{G}$ and $\boldsymbol{R}\pi_*\gamma_!\gamma^{-1}\mathcal{G}[1]$ have homology concentrated in degree 0.*

Moreover, under the equivalences of categories

$$\mathtt{Loc}(L) \simeq \mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(S)] \otimes k[\mu^{\pm}]) \qquad \textit{and} \qquad \mathtt{Loc}(S) \simeq \mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(S)])$$

the functor (25.7.15) *sends a module $V \in \mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(S)] \otimes k[\mu^{\pm}])$ to the map $[V \xrightarrow{\mu-1} V]$ of $k[\pi_1(S)]$-modules.*

Now we get back to the situation when $S = M$ is a submanifold of codimension 2 of a manifold N. Then there is, up to isotopy, a canonical inclusion from the unit normal bundle $L := S_M(N)$ to the complement $U := N \setminus M$ of M in N. We denote this inclusion by $i_L : L \to U$. The normal framing Φ of $M \subset N$ gives a trivialization of the bundle $\pi : L \to S$. This allows us to define the set $K \subset L$ as above, and hence the functor (25.7.15). Using this construction, we define (25.7.12) as the composition

$$T \,:\, \mathtt{Loc}(U) \xrightarrow{i_L^{-1}} \mathtt{Loc}(L) \xrightarrow{\text{(25.7.15)}} \mathrm{Mor}(\,\mathtt{Loc}(S)\,) \tag{25.7.17}$$

Proposition 25.7.16 then allows us to give a simple description of this functor. Retain the notation in the paragraph preceding Theorem 25.7.10. Then under the equivalence

$$\mathtt{Loc}(U) \simeq \mathtt{Loc}(X) \simeq \mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(X)])\,, \qquad \mathtt{Loc}(M) \simeq \mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(M_1)])\times\ldots\times\mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(M_1)]$$

the functor (25.7.17) sends a module V over $k[\pi_1(X)]$ to the map of modules

$$\Big[\,(V,\ldots,V) \xrightarrow{(\,\phi_1(\mu_1)-1\,,\ldots,\,\phi_r(\mu_r)-1\,)} (V,\ldots,V)\,\Big] \quad \in \mathrm{Mor}\,\big(\mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(M_1)]) \times \ldots \times \mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(M_1)]$$

By the first part of Propsition 25.7.16, the closed subset $K \subset L$ is a perverse link bundle in the sense of [118, Definition 4.1]. Therefore, one can apply [118, Theorem 4.5] to show that the category $\mathsf{Perv}(N, M)$ of perverse sheaves on N constructible with respect to the stratification $\{M\,,\, N\setminus M\}$ is equivalent to the gluing category $\mathcal{M}(U, S; T)$ defined above.

By the above description of the functor T, objects of this gluing category can be described as consisting of a tuple $(V, W_1, \ldots W_r)$, where $V \in \mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(X)])$ and $W_i \in \mathtt{Mod}^{\mathrm{fd}}(k[\pi_1(M_i)])$, together with maps $a_i^* : V \to W_i$ and $a_i : W_i \to V$ of $k[\pi_1(M_i)]$-modules, such that $a_i a_i^* = \phi_i(\mu_i) - 1$. (Here, V is regarded as a

$k[\pi_1(M_i)]$-module by the map $k[\pi_1(M_i)] \to k[\pi_1(M_i \times S^1)] \xrightarrow{\phi_i} k[\pi_1(X)]$.) In other words, the gluing category is equivalent to the category of finite dimensional left modules over the k-category

$$\left[\begin{tikzcd} \boxed{k[\pi_1(M_1)]} \arrow[dr, bend left, "a_1"] & \cdots & \boxed{k[\pi_1(M_r)]} \arrow[dl, bend left, "a_r"] \\ & \boxed{k[\pi_1(X)]} \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_r^*"] & \end{tikzcd}\right] \Bigg/ I_0 \tag{25.7.18}$$

where I_0 is the ideal generated by

$$\phi_i(\mu_i) = a_i a_i^* + 1$$

$$\phi_i(\lambda_i) a_i = a_i \lambda_i \quad \text{for all } \lambda_i \in \pi_1(M_i)$$

$$\phi_i(\lambda_i) a_i^* = a_i^* \phi(\lambda_i) \quad \text{for all } \lambda_i \in \pi_1(M_i)$$

In this last k-category, the map $a_i^* a_i + 1$ is invertible, and commute with all the elements in $\pi_1(M_i)$. In fact, a direct calculation shows that its inverse is given by $(a_i^* a_i + 1)^{-1} = 1 - a_i^* \phi_i(\mu)^{-1} a_i$. Since the k-category described in Theorem 25.7.10 is obtained from (25.7.18) by formally adding an invertible variable μ_i commuting with $\pi_1(M_i)$, modulo the relation $\mu_i = a_i^* a_i + 1$, this does not change the isomorphism type of the k-categories. This completes the proof of Theorem 25.7.11.

Remark 25.7.19. In the construction of the perverse neighborhood (Definition 25.4.3) of M, we started with the directed cylinder (25.4.1) of the Adams-Hilton model $\mathsf{AH}(M)$ of M. One could replace the directed cylinder with the l-shifted directed cylinder. *i.e.*, take $\mathcal{B} := \mathsf{AH}(M) \otimes \vec{I}^{[l]}$ in (25.4.1). We expect that the resulting (l-shifted) perversely thickened DG category will be related to perverse sheaves of other perversities.

CHAPTER 26

CONCLUDING REMARKS

We mention some possible relation with other work in the literature. We hope to clarify these relations in future work.

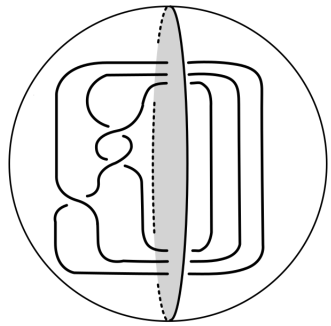
1. In [11], the link DG category $\mathscr{A}(\mathbb{R}^3, L; \Phi)$ was constructed as a different homotopy pushout. Namely, consider the braid group B_n acting on the 2-dimensional disk with n mark points as the mapping class group. This induces an action on the perversely thickened DG category $\mathscr{A}(D^2, \{p_1, \ldots, p_n\})$. In the case for points in the disk, we have seen in Section 25.5 that the DG category has homology concentrated in degree 0. One can therefore consider the braid group action on $\tilde{A}^{(n)} := \mathrm{H}_0((D^2, \{p_1, \ldots, p_n\}))$. This action was studied in [66], which gave explicit formula for the action. It turns out (see [11]) that the perversely thickened DG category for $(\mathbb{R}^3, L)$ is given by the homotopy pushout

$$\mathscr{A}(\mathbb{R}^3, L; \Phi) \;=\; \operatorname{hocolim} \big[\, \tilde{A}^{(n)} \xleftarrow{(\beta,\mathrm{id})} \tilde{A}^{(2n)} \xrightarrow{(\mathrm{id},\mathrm{id})} \tilde{A}^{(n)} \,\big] \tag{26.0.1}$$

where $\beta \in B_n$ is any braid that closes to the link L.

This pushout can be heuristically viewed as a gluing construction. To see this, suppose the braid β is located in the region $\{x < 0\}$, and is closed to the link L by letting the two ends of the braid pass through the hyperplane $\{x = 0\}$

and close in the region $\{x > 0\}$, and in the following diagram



The hyperplane $X_0 = \{x = 0\}$ separates the solid ball B^3 into two regions $X_{\geq 0} = \{x \geq 0\}$ and $X_{\leq 0} = \{x \leq 0\}$. Both of them contain a closed subset of the link L. The picture shows that both of the pair $(X_{\geq 0}, L \cap X_{\geq 0})$ and $(X_{\leq 0}, L \cap X_{\leq 0})$ are homeomorphic to $(D^2, \{p_1, \ldots, p_n\}) \times [0, 1]$. Similarly, the pair $(X_0, L \cap X_0)$ is homeomorphic to $(D^2, \{p_1, \ldots, p_n, p'_n, \ldots, p'_1\})$.

Heuristically, we think of the pair $(D^2, \{p_1, \ldots, p_n\}) \times [0, 1]$ as 'essentially equivalent' to the pair $(D^2, \{p_1, \ldots, p_n\})$, and hence associate to both $(X_{\geq 0}, L \cap X_{\geq 0})$ and $(X_{\leq 0}, L \cap X_{\leq 0})$ the perversely thickened DG category $\tilde{A}^{(n)}$. Under this association, the gluing of the above diagram could be expressed as the homotopy pushout (26.0.1).

This heuristic argument suggests that perversely thickened DG categories should well-behaved under gluing. In other words, they should form a costack of DG categories on stratified spaces. However, to make the statement precise, one should associate a DG category to both of the pairs $(X_{\geq 0}, L \cap X_{\geq 0})$ and $(X_{\leq 0}, L \cap X_{\leq 0})$. These pairs are of the form $(N \times I, M \times I)$, where $M \subset N$ is an embedded manifold. Both $N \times I$ and $M \times I$ are themselves manifolds with boundary. In the gluing constructions, one will then encounter manifolds with corners.

Therefore, even in this simple case, it seems to be useful to generalize the notion of relative Calabi-Yau structure to a "fully extended" one, which covers also manifolds with corners. We expect that the generalization of the perversely deformed DG category to these contexts will have application to perverse sheaves with respect to stratifications other than the simple ones of the form $(M, N \setminus M)$ that we consider in this thesis.

2. By construction, the perverse peripheral pair (25.4.7) always has a relative Calabi-Yau structure (see Theorem 25.4.8). By the main result of [16], announced in [15], this induces shifted Lagrangian structures on the corresponding maps of the derived moduli stacks of pseudo-perfect modules.

V. Shende and A. Takeda obtained a similar result in [144]. Building on the work of D. Nadler [127, 128] on combinatorial models, called *arboreal spaces*, of categories of microlocal sheaves. For a manifold M, together with a Legendrian submanifold $\Lambda \subset ST^*(M)$ in the unit cotangent bundle of M, they studied the sheaf $\mu\mathrm{loc}$ of DG categories of microlocal sheaves on the stratified space $\mathbb{X}$ defined as the union of the zero section $M \subset T^*M$ with the cone in T^*M over Λ. They showed that the pair $(\mathbb{X}, \mu\mathrm{loc})$ forms a so-called locally arboreal space, endowed with a natural orientation structure. The orientation structure then induces shifted Lagrangian structures on the map $\mathcal{M}(\mathbb{X}) \to \mathcal{M}(\partial\mathbb{X})$ of derived moduli stack of pseudo-perfect objects induced by the boundary inclusion $\partial\mathbb{X} \subset \mathbb{X}$.

In particular, they mentioned that the map from the augmentation variety of knot contact homology to $k^\times \times k^\times$ can be identified with the map $\mathcal{M}(\mathbb{X}) \to \mathcal{M}(\partial\mathbb{X})$ in the case when the Legendrian Λ is given by the unit conormal bundle

$\Lambda = ST_L^*(\mathbb{R}^3)$ of the given link L. Therefore this map has a canonical Lagrangian structure.

In view of Theorem 25.6.25, 25.4.8 and 19.0.10, these results are therefore very similar to the results in the present thesis. However, it seems that the approach in the present thesis is more akin to perverse sheaves rather than constructible sheaves in general. (See, for example, Theorem 19.0.10). In particular, it is not clear to us what is the role of t-structures in the work of [125, 126, 127, 128, 144]. It would be very interesting to find a precise relation between *loc.cit.* and the present work.

3. Recent work [38] of K. Cieliebak, T. Ekholm, J. Latschev and L. Ng used string topology operations to define a string homology $\mathrm{H}_*^{\mathrm{string}}(K)$ associated to a codimension 2 submanifold K inside a smooth n-dimensional manifold Q. For the case when $Q = \mathbb{R}^3$ and K is a knot in it, it is proved in [38] that the degree 0 string homology is isomorphic to the degree 0 knot contact homology.

By the work [33] of Cohen and Ganatra (see also [19]), one can view string topology operations as structures induced by an underlying Calabi-Yau strucure on a DG category. From this perspective, it seems useful to clarify whether the string topology operations defining the string homology in [38] can be interpreted algebraically. In particular, we hope that this clarification would lead to relations between [38] and the present work.

4. Throughout Part IV of this thesis, we have considered DG categories, *i.e.*, categories enriched over the symmetric monoidal model category $\mathcal{C}(k)$ of chain complexes over k. However, in all our main constructions, we have only used some very basic properties of shifts and duality on $\mathcal{C}(k)$. Therefore, it seems that there is a direct generalization to categories enriched over other (stable) symmetric monoidal model categories [79], such as the category of spectra (see, e.g., [154]). We expect that such generalization would give spectral enrichments of categories arising in contact geometry. See also [129] for work in this direction.

APPENDIX A

SOME RESULTS ABOUT HOMOTOPY PUSHOUT OF DG CATEGORIES

We collect some results about homotopy pushout of DG categories. These results are probably well-known to experts. However, we have not been able to find an explicit reference. Then, we will use these results to prove that certain explicitly defined DG functors are quasi-equivalences.

Throughout this thesis, when we speak of *homotopy pushout*, we always mean the total left derived functor of the pushout functor (see, e.g., [47] for an overview). In the first part of this appendix, we will give some criteria for a diagram to be adapted under pushout, in the sense of the following

Definition A.0.1. A diagram $[\, X \xleftarrow{f} Z \xrightarrow{g} Y \,]$ in a model category $\mathcal{C}$ is said to be *adapted under pushout* if the canonical mapfrom its homotopy pushout to its ordinary pushout is an isomorphism in the homotopy category $\mathrm{Ho}(\mathcal{C})$.

Recall that a model category $\mathcal{C}$ is said to be *left proper* if pushout of a weak equivalence along a cofibration is still a weak equivalence. If $\mathcal{C}$ is left proper, then $[\, X \xleftarrow{f} Z \xrightarrow{g} Y \,]$ is adapted under pushout if at least one of the maps f or g is a cofibration. However, the category $\mathtt{dgCat}_k$ of all small DG categories, endowed with Tabuada's quasi-equivalence model structure [152], is not left proper if k is not assumed to be a field. The first part of this appendix gives a substitute for left properness that is sufficient for our purposes.

Definition A.0.2. Let $\mathcal{C}$ be a model category. An object $A \in \mathcal{C}$ is said to be *left proper* if, for all trivial fibration $p_A : Q_A \overset{\sim}{\twoheadrightarrow} A$ from a cofibrant object Q_A, and all cofibrations $f : Q_A \hookrightarrow R$, the pushout of p_A along f is a weak equivalence:

$$A \amalg_{Q_A} R \xrightarrow{\sim} R$$

Clearly, if $\mathcal{C}$ is left proper in the ordinary sense, then every object is left proper. We will see that the converse is also true (see Corollary A.0.7), hence no confusion should arise.

We first prove the following

Proposition A.0.3. *Suppose that in the diagram* $P = [\, X \overset{f}{\leftarrow} Z \overset{g}{\hookrightarrow} Y \,]$*, the object* Z *is cofibrant, the map* $g : Z \hookrightarrow Y$ *is a cofibration, and the object* X *is left proper, then* P *is adapted under pushouts.*

Proof. Factorize the map $f : Z \to X$ into $Z \overset{\tilde{f}}{\hookrightarrow} \tilde{X} \overset{\sim}{\twoheadrightarrow} X$. Consider the following two-step pushout

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & \tilde{X} \arrow[r, two heads, "\sim"] \arrow[d, hook] & X \arrow[d, hook] \\ Y \arrow[r, hook] & \tilde{W} \arrow[r] & W \end{tikzcd}$$

where W and $\tilde{W}$ are defined to be the appropriate pushouts. Then, by definition of left properness of X, the map $\tilde{W} \to W$ is a weak equivalence. Since $\tilde{W}$ represents the homotopy pushout of the diagram P, this completes the proof. $\square$

Proposition A.0.4. *Suppose that in the diagram* $P = [\, X \overset{f}{\leftarrow} Z \overset{g}{\hookrightarrow} Y \,]$*, both* X *and* Z *are left proper objects, and the map* $g : Z \hookrightarrow Y$ *is a cofibration, then* P *is adapted under pushouts.*

Proof. Choose replacement $\tilde{P} = [\, X \leftarrow Q_Z \hookrightarrow Q_Y \,]$ of the diagram P. Then Proposition A.0.3 implies that the homotopy colimit of $\tilde{P}$, and hence of P, is represented by the pushout $\tilde{W} := \operatorname{colim}(\tilde{P})$.

Next, notice that the map $Q_Z \to X$ can be factorized as a composition $Q_Z \overset{\sim}{\twoheadrightarrow}$

$Z \xrightarrow{f} X$. Therefore, $\tilde{W}$ fits into the following two-step pushout diagram.

$$\begin{tikzcd} Q_Z \arrow[r, two heads, "\sim"] \arrow[d, hook] & Z \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Q_Y \arrow[r, hook] & \tilde{Y} \arrow[r] & \tilde{W} \end{tikzcd}$$

Then, notice that, by left properness of Z, the map $Q_Y \to \tilde{Y}$ is a weak equivalence, hence so is the canonical map $\tilde{Y} \to Y$.

Therefore, if we let $\tilde{P}' = [\, X \leftarrow Z \hookrightarrow \tilde{Y}\,]$, then the above two-step pushout shows that $\operatorname{colim}(\, \tilde{P}'\,) = \tilde{W}$. Moreover, the canonical map $\tilde{P}' \to P$, which we have seen to be objectwise weak equivalences, induce the canonical map $\tilde{W} \to W$, which we have seen to represent the map from the homotopy pushout of P to the ordinary pushout of P. The proof is therefore complete by applying the following lemma. $\square$

Lemma A.0.5. *Let $\mathcal{C}$ be any model category. Suppose $Z \hookrightarrow \tilde{Y}$ is a cofibration, and $\tilde{Y} \xrightarrow{\sim} Y$ is a weak equivalence such that the composition $Z \hookrightarrow \tilde{Y} \to Y$ is a cofibration. Then pushout along any map $Z \to X$ induces a weak equivalence $X \amalg_Z \tilde{Y} \to X \amalg_Z \tilde{Y}$.*

Proof. Let $Z \downarrow \mathcal{C}$ and $X \downarrow \mathcal{C}$ be the under categories of Z and X respectively. Then pushout determines a functor

$$X \amalg_Z - : Z \downarrow \mathcal{C} \to X \downarrow \mathcal{C} \tag{A.0.6}$$

This functor has a right adjoint defined by pre-composing with $Z \to X$.

This right adjoint clearly preserves weak equivalences and fibrations. Therefore, this adjunction is a Quillen adjunction. In particular, the functor (A.0.6) preserves weak equivalences between cofibrant objects. This completes the proof. $\square$

Corollary A.0.7. *Let $f : Z \xrightarrow{\sim} X$ be a weak equivalence between left proper objects in $\mathcal{C}$, then the pushout of f along any cofibration $Z \hookrightarrow Y$ is a weak equivalence.*

Now, we determine which DG category is left proper when considered to be in the category $\texttt{dgCat}_k$, endowed with Tabuada's quasi-equivalence model structure. A sufficient condition is given by the following

Proposition A.0.8. *If a DG category $\mathcal{A} \in \texttt{dgCat}_k$ is k-flat, then it is left proper.*

Proof. Find a cofibrant resolution $\pi : \mathcal{Q} \overset{\sim}{\twoheadrightarrow} \mathcal{A}$, where $\mathcal{Q}$ is also k-flat. Take I to be the class of generating cofibrations in [152]. Since every cofibration is a retract of a relative I-cell complexes (see [77]), and since quasi-equivalences are preserved under direct limits, it suffices to consider pushouts of π along single I-cell attachments.

There are two types of such attachments. The first is given by taking the disjoint union with a point, *i.e.*, $F : \mathcal{Q} \hookrightarrow \mathcal{Q} \amalg *$. (Here, $*$ refers to the DG category with one object whose endomorphism ring is k.) In this case, the pushout of $\pi : \mathcal{Q} \to \mathcal{A}$ along F is given by $\mathcal{Q} \amalg * \to \mathcal{A} \amalg *$, which is clearly a quasi-equivalence.

The second type is obtained by attaching a generating arrow f to $\mathcal{Q}$. We write this as $F : \mathcal{Q} \to \mathcal{Q}\langle f \rangle$ The corresponding pushout of π along F is then given by $\tilde{\pi} : \mathcal{Q}\langle f \rangle \to \mathcal{A}\langle f \rangle$. Suppose that the arrow points from $x \in \mathrm{Ob}(\mathcal{Q})$ to $y \in \mathrm{Ob}(\mathcal{Q})$. By an abuse of notation, denote the image $\pi(w) \in \mathrm{Ob}(\mathcal{A})$ of an object $w \in \mathrm{Ob}(\mathcal{Q})$ still as w. Then for each $w, z \in \mathrm{Ob}(\mathcal{Q})$, we have

$$\mathcal{Q}\langle f \rangle(w, z) = \mathcal{Q}(w, z) \oplus \bigoplus_{n \geq 1} \mathcal{Q}(y, z) \cdot f \cdot \mathcal{Q}(y, x) \cdots \mathcal{Q}(y, x) \cdot f \cdot \mathcal{Q}(w, x)$$

Denoting each summand by $\mathcal{Q}\langle f\rangle(w,z)^{(n)}$, then we have

$$\mathcal{Q}\langle f\rangle(w,z)^{(0)} = \mathcal{Q}(w,z) \qquad\qquad \mathcal{Q}\langle f\rangle(w,z)^{(n)} \cong \mathcal{Q}(y,z)\otimes\mathcal{Q}(y,x)^{\otimes(n-1)}\otimes\mathcal{Q}(w,x)$$

Moreover, if we let $\mathcal{Q}\langle f\rangle(w,z)^{(\leq n)}$ to be the sum of all summands from 0 to n, then this forms a filtration that is preserves by the differential.

The same description also holds for $\mathcal{A}\langle f\rangle(w,z)$. The canonical map $\mathcal{Q}\langle f\rangle(w,z) \to \mathcal{A}\langle f\rangle(w,z)$ preserves this filtration. By k-flatness of both $\mathcal{A}$ and $\mathcal{Q}$, this map induces an isomorphism on the E_2-page of the associated spectral sequence. Since these filtrations are bounded below, this completes the proof. □

Remark A.0.9. Proposition A.0.4 and A.0.8 can be combined to show that if $\mathcal{P} := [\,\mathcal{A}_1 \xleftarrow{G} \mathcal{A}_0 \xrightarrow{F} \mathcal{A}_2\,]$ is a diagram of k-flat DG categories and if either F or G is a cofibration, then the homotopy pushout of $\mathcal{P}$ is represented by the ordinary pushout of $\mathcal{P}$.

We will also encounter situations where, say, F is not a cofibration, but a certain extension $(F,F') : \mathcal{A}_0 \amalg \mathcal{A}_0' \hookrightarrow \mathcal{A}_2$ is a cofibration. In this situtation, form the diagram

$$\mathcal{P}' := [\,\mathcal{A}_1 \amalg \mathcal{A}_0' \xleftarrow{G\amalg\mathrm{id}} \mathcal{A}_0 \amalg \mathcal{A}_0' \xrightarrow{(F,F')} \mathcal{A}_2\,]$$

Then it is easy to see that the pushout of $\mathcal{P}'$ equals the pushout of $\mathcal{P}$. The same is true for homotopy pushouts. This again shows that $\mathcal{P}$ is adapted under pushout.

Now we consider homotopy inversion of degree zero closed morphisms. Recall that the directed interval $\vec{I}$ is the free DG category over the quiver $[\,\bullet \xrightarrow{v} \bullet\,]$, while the (undirected) interval I is the localization $I = \vec{I}[v^{-1}]$ (see the beginning of Section 24.2). Specifying a degree zero closed morphism $f \in \mathcal{A}(x,y)_0$ in a DG category $\mathcal{A}$ is equivalent to giving a DG functor $\hat{f} : \vec{I} \to \mathcal{A}$. The DG functor $\hat{f}$ pass to a DG functor from I if and only if f is strictly invertible (*i.e.,* invertible in the underlying category $\mathrm{Z}_0(\mathcal{A})$). Thus, inverting the morphism f is equivalent to taking the pushout $\mathcal{A}[f^{-1}] = \operatorname{colim}[\, I \leftarrow \vec{I} \xrightarrow{\hat{f}} \mathcal{A}\,]$. For this reason, we call the following homotopy pushout the *homotopy inversion* of f

$$\mathcal{A}\langle f^{-1}\rangle := \operatorname{hocolim}\left[I \leftarrow \vec{I} \xrightarrow{\hat{f}} \mathcal{A} \right] \tag{A.0.10}$$

The proposition below concerns homotopy inversion of a degree zero morphism in a non-negatively graded DG category. In the language of [107, Appendix A.3.2], this proposition amounts to saying that the symmetric monoidal model category of non-negatively graded chain complexes satisfies the 'invertibility hypothesis'. However, throughout [107, Appendix A.3], Lurie imposed assumptions on a symmetric monoidal model category **S** that guarantee left properness of the model category of **S**-enriched categories. Since the category of (non-negatively graded) DG categories is in general not left proper if k is not assumed to be a field, we cannot use the results in *loc.cit.* directly. Instead, we reproduce the following proposition, whose proof is parallel to that of [107, Lemma A.3.2.20].

Proposition A.0.11. *Let $\mathcal{A}$ be a non-negatively graded DG category. Suppose $f \in \mathcal{A}(x,y)_0$ is a degree 0 morphism in $\mathcal{A}$ whose image in $\mathrm{H}_0(\mathcal{A})$ is an invertible morphism, then the canonical map $\mathcal{A} \to \mathcal{A}\langle f^{-1}\rangle$ is a quasi-equivalence.*

Proof. By the Quillen equivalence [153] between the category $\mathtt{dgCat}_{\geq 0}$ of non-negatively graded DG categories and the category $\mathtt{Cat}_{\mathtt{sMod}_k}$ of categories enriched over simplicial k-modules, it suffices to prove the corresponding statement in $\mathtt{Cat}_{\mathtt{sMod}_k}$. Consider the Quillen adjunction [153]

$$F = k[-] \,:\, \mathtt{Cat}_\Delta \rightleftarrows \mathtt{Cat}_{\mathtt{sMod}_k} \,:\, G$$

between $\mathtt{Cat}_{\mathtt{sMod}_k}$ and the category $\mathtt{Cat}_\Delta$ of simplicially enriched categories, endowed with Bergner's model structure [18].

Let $\vec{I}_\Delta$ be the (poset) category $[\bullet \xrightarrow{v} \bullet]$, and let I_Δ be its localization $I_\Delta[v^{-1}]$. Then we have $\vec{I} = F(\vec{I}_\Delta)$ and $I = F(I_\Delta)$. Like in the case of DG categories, for any $\mathcal{P} \in \mathtt{Cat}_\Delta$, and any degree zero morphism $g \in \mathcal{P}(x,y)$ classified by a simplicial functor $\hat{g} : \vec{I}_\Delta \to \mathcal{P}$, we denote by $\mathcal{P}\langle g^{-1}\rangle$ the homotopy pushout

$$\mathcal{P}\langle g^{-1}\rangle \;:=\; \operatorname{hocolim}\,[\, I_\Delta \leftarrow \vec{I}_\Delta \xrightarrow{\hat{g}} \mathcal{P}\,]$$

taken in the model category $\mathtt{Cat}_\Delta$.

It is proved in [49] that if g is invertible in $\pi_0(\mathcal{P})$, then the canonical map $\mathcal{P} \to \mathcal{P}\langle g^{-1}\rangle$ is a weak equivalence in $\mathtt{Cat}_\Delta$. We call this the invertibility hypothesis for simplicial sets. (See also [107, Example A.3.2.18].)

Now, suppose we are given a degree zero morphism $f \in \mathcal{A}(x,y)_0$ in a category $\mathcal{A}$ enriched over simplicial modules, such that f is invertible in $\pi_0(\mathcal{A})$, then let $g \in G(\mathcal{A})(x,y)_0$ be the same morphism, considered in $G(\mathcal{A}) \in \mathtt{Cat}_\Delta$. By the above mentioned invertibility hypothesis for simplicial sets, the canonical map $G(\mathcal{A}) \to G(\mathcal{A})\langle g^{-1}\rangle$ is a weak equivalence.

Notice that the simplicial functor $\hat{g} : \vec{I}_\Delta \to G(\mathcal{A})$ classifing by g is adjoint to the simplicial k-linear functor $\hat{f} : \vec{I} = F(\vec{I}_\Delta) \to \mathcal{A}$ classifying f. Therefore, the composition $F(\vec{I}_\Delta) \xrightarrow{F(\hat{g})} FG(\mathcal{A}) \xrightarrow{\epsilon_{\mathcal{A}}} \mathcal{A}$ is the map $\hat{f} : \vec{I} \to \mathcal{A}$. Thus, we have the following two-step homotopy pushout

$$\begin{tikzcd} F(\vec{I}_\Delta) \arrow[r, "F(\hat{g})"] \arrow[d] & FG(\mathcal{A}) \arrow[r, "\epsilon_{\mathcal{A}}"] \arrow[d, "\sim"] & \mathcal{A} \arrow[d] \\ F(I_\Delta) \arrow[r] & F\left[G(\mathcal{A})\langle g^{-1}\rangle\right] \arrow[r] & \mathcal{A}\langle f^{-1}\rangle \end{tikzcd}$$

We consider this as a commutative diagram in $\mathrm{Ho}(\mathtt{Cat}_{\mathtt{sMod}_k})$. Since F is exact and left Quillen, the left hand side square represents a homotopy pushout. The composition of the two squares is, by definition, also a homotopy pushout. Therefore, the right hand side square is also a homotopy pushout. However, the vertical map $FG(\mathcal{A}) \to F\left[G(\mathcal{A})\langle g^{-1}\rangle\right]$ is a weak equivalence by the invertibility hypothesis for simplicial sets. Hence so is its homotopy pushout $\mathcal{A} \to \mathcal{A}\langle f^{-1}\rangle$. This completes the proof. $\square$

Now we will apply these general statements to prove that certain DG functors are quasi-equivalences. Let $A_1, \ldots, A_n$ be DG algebras. Consider the DG category presented by

$$\mathcal{A}_1 = \left[\begin{tikzcd} \boxed{A_1} \arrow[loop above, "\xi_1"] \arrow[dr, bend left, "a_1"] & \cdots & \boxed{A_n} \arrow[loop above, "\xi_n"] \arrow[dl, bend left, "a_n"] \\ & \bullet \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_n^*"] & \end{tikzcd}\right] \tag{A.0.12}$$

where the generating arrows $a_1, \ldots, a_n, a_1^*, \ldots, a_n^*$ are degrees 0 closed elements, and the arrows $\xi_1, \ldots, \xi_n$ have degree 1, with differential $d(\xi_i) = z_i - a_i^* a_i$, for some degree 0 closed elements $z_i \in A_i$.

There is a DG functor $\pi_1 : \mathcal{A}_1 \to \mathcal{A}_2$ to the following DG category

$$\mathcal{A}_2 = \left[\begin{tikzcd} \boxed{A_1} \arrow[dr, bend left, "a_1"] & \cdots & \boxed{A_n} \arrow[dl, bend left, "a_n"] \\ & \bullet \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_n^*"] & \end{tikzcd}\right] \Big/ (z_i - a_i^* a_i) \tag{A.0.13}$$

sending the generators ξ_i to 0.

Proposition A.0.14. *The map $\pi_1 : \mathcal{A}_1 \to \mathcal{A}_2$ is a quasi-equivalence.*

Proof. We will give a description of the Hom complexes of the DG category $\mathcal{A}_1$. There are four kinds of Hom complexes: $\mathcal{A}_1(i,j)$, $\mathcal{A}_1(i,0)$, $\mathcal{A}_1(0,j)$, and $\mathcal{A}_1(0,0)$. All of them have a similar description. We consider the case $\mathcal{A}_1(i,j)$ first.

For $r \geq 0$, let S_r be the set of all sequences $(i_1, \dots, i_r)$ of integers, where $i_j \in \{1, \dots, n\}$ for all $1 \leq j \leq r$, and $i_j \neq i_{j+1}$ for $1 \leq j \leq r-1$. Moreover, for $j \in \{1, \dots, n\}$, denote by $S_r^j \subset S_r$ the subset consisting of tuples $(i_1, \dots, i_r)$ such that $i_r \neq j$. Similarly, denote by ${}^iS_r \subset S_r$ those such that $i_1 \neq i$. Let ${}^iS_r^j := {}^iS_r \cap S_r^j$.

Then the Hom complex $\mathcal{A}_1(i,j)$ has a description

$$\mathcal{A}_1(i,j) = \bigoplus_{r \geq 0} \bigoplus_{(i_1,\dots,i_r) \in {}^iS_r^j} \tilde{A}_j \cdot a_j^* a_{i_r} \cdot \tilde{A}_{i_r} \cdots \tilde{A}_{i_1} \cdot a_{i_1}^* a_i \cdot \tilde{A}_i \tag{A.0.15}$$

where $\tilde{A}_l$ is the DG algebra $\tilde{A}_l := A_l \langle a_l^* a_l, \xi_l \rangle$ obtained by freely adjoining the symbols ξ_l and $a_l^* a_l$, with differentials $d(a_l^* a_l) = 0$ and $d(\xi_l) = z_l - a_l^* a_l$.

Alternatively, the elements $w_l = z_l - a_l^* a_l$ and ξ_l also form a set of semi-free generators of $\tilde{A}_l$ over A_l. Therefore, we have $\tilde{A}_l = A_l * k\langle w_l, z_l \rangle$. Now, if we pass to the quotient $\pi_1 : \mathcal{A}_1 \to \mathcal{A}_2$ by modding out the ideals generated by the elements $z_l - a_l^* a_l$ and ξ_l, then we simply remove all the occurrences of the symbols w_l and ξ_l in the description (A.0.15). Therefore, the Hom complexes $\mathcal{A}_2(i,j)$ have a description

$$\mathcal{A}_2(i,j) = \bigoplus_{r \geq 0} \bigoplus_{(i_1,\dots,i_r) \in {}^iS_r^j} A_j \cdot a_j^* a_{i_r} \cdot A_{i_r} \cdots A_{i_1} \cdot a_{i_1}^* a_i \cdot A_i \tag{A.0.16}$$

Thus, the proof that $\mathcal{A}_1(i,j) \to \mathcal{A}_2(i,j)$ is a quasi-isomorphism is complete once we prove that, for each $(i_1, \ldots, i_r) \in {}^iS_r^j$, the map

$$\tilde{A}_j \otimes \tilde{A}_{i_r} \otimes \ldots \otimes \tilde{A}_{i_1} \otimes \tilde{A}_i \; \to \; A_j \otimes A_{i_r} \otimes \ldots \otimes A_{i_1} \otimes A_i \tag{A.0.17}$$

is a quasi-isomorphism.

Let V be the complex of k-modules $[\, 0 \;\to\; k \;\xrightarrow{\mathrm{id}}\; k \;\to\; 0\,]$ concentrated in degree 0 and 1. Then it is easy to see that the DG algebra $\tilde{A}_l$ can be described by

$$\tilde{A}_l \quad \cong \quad A_l \quad \oplus \quad A_l \otimes V \otimes A_l \quad \oplus \quad A_l \otimes V \otimes A_l \otimes V \otimes A_l \quad \oplus \quad \ldots$$

where each summand is a chain subcomplex. Since V is null-homotopic as a chain complex, so is each component in this description, except the first component A_l. Therefore, the canonical map $\tilde{A}_l \;\to\; A_l$ is a homotopy equivalence as a map of chain complexes. Since homotopy equivalences are preserved under tensor products, this completes the proof that the map $\mathcal{A}_1(i, j) \;\to\; \mathcal{A}_2(i, j)$ is a quasi-isomorphism.

For the other types of chain complexes, the proof also follows the same line of argument starting from the following analogue of the description (A.0.15).

$$\begin{aligned}
\mathcal{A}_1(i, 0) \;&=\; \bigoplus_{r \geq 0} \; \bigoplus_{(i_1, \ldots, i_r) \in {}^iS_r} \quad a_{i_r} \cdot \tilde{A}_{i_r} \cdots \tilde{A}_{i_1} \cdot a_{i_1}^* a_i \cdot \tilde{A}_i \\
\mathcal{A}_1(0, j) \;&=\; \bigoplus_{r \geq 0} \; \bigoplus_{(i_1, \ldots, i_r) \in S_r^j} \quad \tilde{A}_j \cdot a_j^* a_{i_r} \cdot \tilde{A}_{i_r} \cdots \tilde{A}_{i_1} \cdot a_{i_1}^* \\
\mathcal{A}_1(0, 0) \;&=\; k \,\oplus\, \bigoplus_{r \geq 0} \; \bigoplus_{(i_1, \ldots, i_r) \in S_r} \quad a_{i_r} \cdot \tilde{A}_{i_r} \cdots \tilde{A}_{i_1} \cdot a_{i_1}^*
\end{aligned} \tag{A.0.18}$$

One has a similar description for these Hom complexes of $\mathcal{A}_2$, by replacing $\tilde{A}_l$ by A_l. The same argument as above then shows that the canonical map $\pi_1 \,:\, \mathcal{A}_1 \;\to\; \mathcal{A}_2$ induces quasi-isomorphisms on these chain complexes. This completes the proof. $\square$

Next, we consider the following two DG categories.

$$\mathcal{A}_3 = \left[\begin{tikzcd} \boxed{A_1} \arrow[loop above, "\xi_1"] \arrow[dr, bend left, "a_1"] & \cdots & \boxed{A_n} \arrow[loop above, "\xi_n"] \arrow[dl, bend left, "a_n"] \\ & \boxed{k\langle T_1,\dots,T_n\rangle} \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_n^*"] \arrow[loop below, "\xi'_1,\dots,\xi'_n"] & \end{tikzcd}\right] \quad \mathcal{A}_4 = \left[\begin{tikzcd} \boxed{A_1} \arrow[loop above, "\xi_1"] \arrow[dr, bend left, "a_1"] & \cdots & \boxed{A_n} \arrow[loop above, "\xi_n"] \arrow[dl, bend left, "a_n"] \\ & \boxed{k\langle T_1^{\pm},\dots,T_n^{\pm}\rangle} \arrow[ul, bend left, "a_1^*"] \arrow[ur, bend left, "a_n^*"] \arrow[loop below, "\xi'_1,\dots,\xi'_n"] & \end{tikzcd}\right] \tag{A.0.19}$$

where the generators T_i are closed of degree 0, and the generators ξ'_i have degree 1, with differentials $d(\xi'_i) = T_i - a_i a_i^* - 1$.

Clearly, these come with a canonical DG functor $\mathcal{A}_3 \to \mathcal{A}_4$. Moreover, there is a DG functor $\mathcal{A}_3 \to \mathcal{A}_1$ sending T_i to $a_i a_i^* + 1$ and ξ'_i to 0.

Assume now that $A_1, \dots, A_n$ are non-negatively graded and k-flat, and suppose that the element $z_i \in A_i$ used to specify the differentials of (A.0.12) are of the form $z_i = \mu_i - 1$ for some invertible elements $\mu_i \in (A_i)_0^{\times}$ of degree 0. Then a direct calculation shows that the element $a_i a_i^* + 1$ is invertible in $\mathcal{A}_2$, with inverse $1 - a_i \mu^{-1} a_i^*$. Therefore, there is a canonical map $\mathcal{A}_4 \to \mathcal{A}_2$ sending T_i to $a_i a_i^* + 1$ and ξ'_i to 0. These maps form a commutative diagram

$$\begin{tikzcd} \mathcal{A}_3 \arrow[r] \arrow[d] & \mathcal{A}_4 \arrow[d] \\ \mathcal{A}_1 \arrow[r] & \mathcal{A}_2 \end{tikzcd} \tag{A.0.20}$$

Proposition A.0.21. *All the maps in the commutative diagram* (A.0.20) *are quasi-equivalences.*

Proof. We have shown that the map $\mathcal{A}_1 \to \mathcal{A}_2$ is a quasi-equivalence. To show that the map $\mathcal{A}_3 \to \mathcal{A}_1$ is a quasi-equivalence, we notice that it is the pushout of the quasi-equivalence $k\langle T_1, \dots, T_n, \xi'_1, \dots, \xi'_n\rangle \to k$ along the map

$F_1 : k\langle T_1, \ldots, T_n, \xi'_1, \ldots, \xi'_n \rangle \to \mathcal{A}_3$. This last map has an extension

$$F : k\langle T_1, \ldots, T_n, \xi'_1, \ldots, \xi'_n \rangle \amalg A_1 \amalg \ldots \amalg A_n \quad \to \quad \mathcal{A}_3$$

that is a semi-free extension, and hence a cofibration. The claim that $\mathcal{A}_3 \to \mathcal{A}_1$ is a quasi-equivalence then follows by applying the trick described in Remark A.0.9.

To prove that $\mathcal{A}_3 \to \mathcal{A}_4$ is a quasi-equivalence, we notice that this map is the homotopy inversion map of the generating arrows $T_1, \ldots, T_n$. *i.e.*, in the notation of (A.0.10), we have $\mathcal{A}_4 = \mathcal{A}_3\langle T_1^{-1}, \ldots, T_n^{-1} \rangle$. Now, as in the calculation preceeding this proposition, the elements T_i are invertible in $\mathrm{H}_0(\mathcal{A}_3)$, with inverse given by $1 - a_i \mu^{-1} a_i^*$. Therefore, an application of Proposition A.0.11 shows that the DG functor $\mathcal{A}_3 \to \mathcal{A}_4$ is a quasi-equivalence. This completes the proof. $\square$

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